EE276: Problem Session #1 Problems

1. **Conceptual.**

   (a) Are quantities like $\mathbb{E}(X)$ and $H(X)$ functions of the random variable X or functions of X's probability mass function or function of something else? Explain. Are the notations used to denote these quantities ideal? If not, suggest alternative notations.

   (b) Do $\mathbb{E}(X)$ and $H(X)$ depend on the particular labels that X takes on (i.e., the alphabet $\mathcal{X}$ of X)?

2. **Entropy of a disjoint mixture.** Let X_1 and X_2 be discrete random variables drawn according to probability mass functions $p_1(\cdot)$ and $p_2(\cdot)$ over the respective alphabets $\mathcal{X}_1 = \{1, 2, \ldots, m\}$ and $\mathcal{X}_2 = \{m+1, \ldots, n\}$. Let

$$X = \begin{cases} X_1, & \text{with probability } \alpha, \\ X_2, & \text{with probability } 1-\alpha. \end{cases}$$

   (a) Find $H(X)$ in terms of $H(X_1)$ and $H(X_2)$ and α.

   (b) Maximize over α to show that $2^{H(X)} \leq 2^{H(X_1)} + 2^{H(X_2)}$ and interpret using the notion that $2^{H(X)}$ is the effective alphabet size.

3. **Log sum inequality.** Prove the log sum inequality: for positive numbers, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, we have

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}. \tag{1}$$

   Use it to directly prove that $D(p||q) \geq 0$.

4. **Coin weighing.** Suppose one has n coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a balance.

   (a) Find an upper bound on the number of coins n so that k weighings will find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.

   (b) *(Difficult)* What is the coin weighing strategy for $k = 3$ weighings and 12 coins?

5. **Markov's inequality for probabilities.** Let $p(x)$ be a probability mass function. Prove, for all $d \geq 0$,

$$\mathbb{P}\left(\{p(X) \leq d\}\right) \log \left(\frac{1}{d} \right) \leq H(X). \tag{2}$$