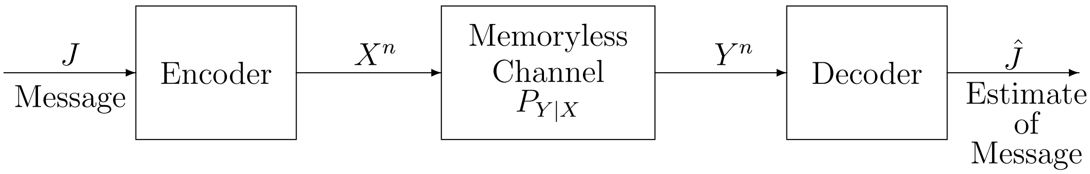
EE276: Homework #5

Due on Friday Feb 14, 6pm - Gradescope entry code: 5K35EZ

1. **Minimizing Channel Probability of Error**.
Below, we are given a communication setting as seen in lecture.

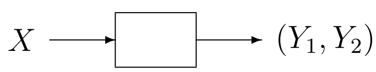


J is a message uniformly distributed on $\{1, 2, ..., M\}$ passed into the system. The encoder maps message J onto its corresponding n-length codeword X^n from codebook $c_n = \{X^n(1), X^n(2), ..., X^n(M)\}$. The encoded message is sent through a memoryless channel characterized by $P_{Y|X}$, and we receive Y^n as output.

The decoder is responsible for estimating J from Y^n; it is a function $\hat{J}$ that maps Y^n to one of the symbols in $\{1, 2, ..., M, \text{error}\}$. We define the probability of error $P_e = P(\hat{J}(Y^n) \neq J)$. Show that P_e, for a fixed codebook c_n, is minimized by:

$$\hat{J}(y^n) = \text{argmax}_{1 \leq j \leq M} P(J = j | Y^n = y^n).$$

2. **The two-look Gaussian channel.**



Consider the ordinary Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1 \quad (1)$$
$$Y_2 = X + Z_2 \quad (2)$$

with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(\mathbf{0}, K)$, where

$$K = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \quad (3)$$

Find the capacity C for

(a) $\rho = 1$

(b) $\rho = 0$

(c) ρ = -1

Hint: The formula for the differential entropy of a Gaussian random vector can be found in the book (refer to Theorem 8.2). It can also be derived based on the formulas for differential entropy and the Gaussian PDF.

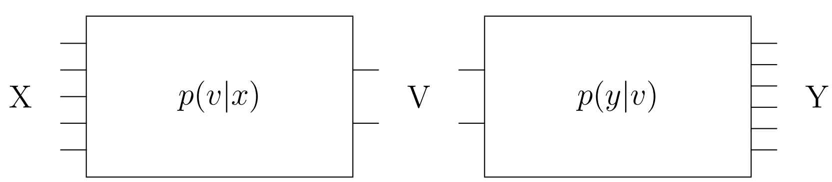
3. **Output power constraint**. Consider an additive white Gaussian noise channel with an expected *output* power constraint P. Thus $Y = X + Z$, $Z \sim N(0, \sigma^2)$, Z is independent of X, and $EY^2 \leq P$. Find the channel capacity.

4. **Gaussian mutual information**
Suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2. Find $I(X; Z)$.

Hint: Refer to Theorem 8.2 in the textbook. In the case of a bivariate normal distribution, $\mathbb{E}[X|Y] = \rho_{X,Y} \frac{\sigma_X}{\sigma_Y} Y$, where $\rho_{X,Y}$ is the correlation coefficient and σ_A is the standard deviation of random variable A.

5. **Bottleneck channel**

Suppose a signal $X \in \mathcal{X} = \{1, 2, \ldots, m\}$ goes through an intervening transition $X \longrightarrow V \longrightarrow Y$:



where $\mathcal{X} = \{1, 2, \ldots, m\}$, $\mathcal{Y} = \{1, 2, \ldots, m\}$, and $\mathcal{V} = \{1, 2, \ldots, k\}$. Here $p(v|x)$ and $p(y|v)$ are arbitrary and the channel has transition probability $p(y|x) = \sum_v p(v|x)p(y|v)$.

Show $C \leq \log k$.

6. **Joint typicality.**
Let (X_i, Y_i, Z_i) be *i.i.d.* according to $p(x, y, z)$. With finite alphabets, we will say that (x^n, y^n, z^n) is jointly typical (written $(x^n, y^n, z^n) \in A_\epsilon^{(n)}$) if

- $p(x^n) \in 2^{-n(H(X)\pm\epsilon)}$
- $p(y^n) \in 2^{-n(H(Y)\pm\epsilon)}$
- $p(z^n) \in 2^{-n(H(Z)\pm\epsilon)}$
- $p(x^n, y^n) \in 2^{-n(H(X,Y)\pm\epsilon)}$
- $p(x^n, z^n) \in 2^{-n(H(X,Z)\pm\epsilon)}$
- $p(y^n, z^n) \in 2^{-n(H(Y,Z)\pm\epsilon)}$
- $p(x^n, y^n, z^n) \in 2^{-n(H(X,Y,Z)\pm\epsilon)}$

Note that $p(a) \in 2^{-n(k \pm \epsilon)}$ means that $p(a) \in \left[2^{-n(k+\epsilon)}, 2^{-n(k-\epsilon)}\right]$.

Now suppose $(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)$ is drawn according to $p(x^n)p(y^n)p(z^n)$. Thus $\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n$ have the same marginals as $p(x^n, y^n, z^n)$ but are independent. Find (bounds on) $Pr\{(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\epsilon^{(n)}\}$ in terms of the entropies $H(X)$, $H(Y)$, $H(Z)$, $H(X,Y)$, $H(X,Z)$, $H(Y,Z)$ and $H(X,Y,Z)$.