EE276: Homework #1

Due on Friday Jan 17, 6pm - Gradescope entry code: 5K35EZ

Note: Mutual information (denoted $I(X;Y)$) will be covered in the Tuesday, Jan 14th lecture. You only need this concept for three sub-parts of this homework.

1. **Example of joint entropy.** Let $p(x, y)$ be given by

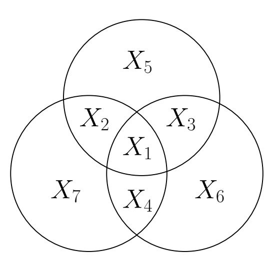
X \ Y	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	0	$\frac{1}{2}$

   Find

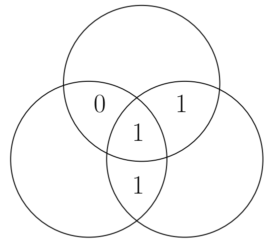
   (a) $H(X), H(Y)$.
   (b) $H(X \mid Y), H(Y \mid X)$.
   (c) $H(X, Y)$.
   (d) $H(Y) - H(Y \mid X)$.
   (e) $I(X;Y)$.
   (f) Draw a Venn diagram for the quantities in (a) through (e).

   Numerically round the answers to three decimal places.

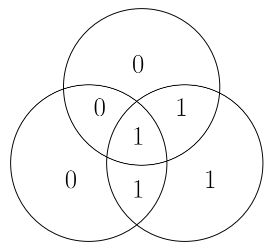
2. **Entropy of Hamming Code.**
   Hamming code is a simple error-correcting code that can correct up to one error in a sequence of bits. Now consider information bits $X_1, X_2, X_3, X_4 \in \{0, 1\}$ chosen uniformly at random, together with check bits X_5, X_6, X_7 chosen to make the parity of the circles even.
   (eg: $X_1 + X_2 + X_4 + X_7 = 0 \mod 2$)



   Thus, for example,



becomes



That is, 1011 becomes 1011010.

(a) What is the entropy $H(X_1, X_2, ..., X_7)$ of $\mathbf{X} := (X_1, \ldots, X_7)$?

Now we make an error (or not) in one of the bits (or none). Let $\mathbf{Y} = \mathbf{X} \oplus \mathbf{e}$, where $\mathbf{e}$ is equally likely to be $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, or $(0, 0, \ldots, 0)$, and $\mathbf{e}$ is independent of $\mathbf{X}$.

(b) Show that one can recover the message $\mathbf{X}$ perfectly from $\mathbf{Y}$. (Please provide a justification, detailed proof not required.)

(c) What is $H(\mathbf{X}|\mathbf{Y})$?

(d) What is $I(\mathbf{X}; \mathbf{Y})$?

(e) What is the entropy of $\mathbf{Y}$?

3. **Entropy of functions of a random variable.** Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned}
H(X, g(X)) &\stackrel{(a)}{=} H(X) + H(g(X) \mid X) &(1)\\
&\stackrel{(b)}{=} H(X); &(2)\\
H(X, g(X)) &\stackrel{(c)}{=} H(g(X)) + H(X \mid g(X)) &(3)\\
&\stackrel{(d)}{\geq} H(g(X)). &(4)
\end{aligned}$$

Thus $H(g(X)) \leq H(X)$.

4. **Coin flips.** A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

   (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

   $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}.$$

   (b) A random variable X is drawn according to this distribution. Construct an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S?" that determine the value of X. Compare $H(X)$ to the expected number of questions required to determine X.

5. **Minimum entropy.** In the following, we use $H(p_1, ..., p_n) \equiv H(\mathbf{p})$ to denote the entropy $H(X)$ of a random variable X with alphabet $\mathcal{X} := \{1, \ldots, n\}$, i.e.,

   $$H(X) = -\sum_{i=1}^{n} p_i \log(p_i).$$

   What is the minimum value of $H(p_1, ..., p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s which achieve this minimum.

6. **Mixing increases entropy.** Let $p_i > 0, \quad i = 1, 2, \ldots m$. Show that the entropy of a random variable distributed according to $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$, is less than the entropy of a random variable distributed according to $(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m)$.

7. **Infinite entropy.** [**Bonus**]
   This problem shows that the entropy of a discrete random variable can be infinite. (In this question you can take log as the natural logarithm for simplicity.)

   (a) Let $A = \sum_{n=2}^{\infty} (n \log^2 n)^{-1}$. Show that A is finite by bounding the infinite sum by the integral of $(x \log^2 x)^{-1}$.

   (b) Show that the integer-valued random variable X distributed as:
   $P(X = n) = (An \log^2 n)^{-1}$ for $n = 2, 3, \ldots$ has entropy $H(X)$ given by:

   $$H(X) = \log A + \sum_{n=2}^{\infty} \frac{1}{An \log n} + \sum_{n=2}^{\infty} \frac{2 \log \log n}{An \log^2 n}$$

   (c) Show that the entropy $H(X) = +\infty$ (by showing that the sum $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges).