Outline

• diagonalizability

• eigen decomposition theorem

• applications (modal forms, asymptotic growth rate)
Diagonalizability

- consider square matrix $A \in \mathbb{R}^{n \times n}$. Assume that $\lambda \in \mathbb{C}$ is an eigenvalue of $A$

- then, $A$ is diagonalizable if and only if the multiplicity of $\lambda_i$ equals $\dim(N(\lambda_i I - A)) = n - \text{rank}(\lambda_i I - A)$, for all $i$
Diagonalizability

- to see this, note that we can write the characteristic polynomial of $A$ as

$$\chi(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \ldots (\lambda - \lambda_k)^{n_k}$$

- the eigenvectors corresponding to $\lambda_i$ are given by the linear system

$$AX = \lambda_i X$$

or $$(A - \lambda_i I)X = 0$$

- so if $\dim N(\lambda_i I - A) = n_i$, then we can find $n_i$ independent eigenvectors corresponding to $\lambda_i$

- if this condition holds for all $i$, then the matrix $A$ is diagonalizable (and vice versa)
Example

is this matrix diagonalizable?

\[
A = \begin{bmatrix}
3 & 1 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4
\end{bmatrix} \in \mathbb{R}^{3 \times 3}
\]

Solution.

- let’s write the characteristic polynomial for \( A \): \( \chi(A) = \text{det}(\lambda I - A) = (\lambda - 4)(\lambda - 3)^2 \)

- eigenvalues are \( \lambda_1 = 4 \) with multiplicity 1, and \( \lambda_2 = 3 \) with multiplicity 2

- the condition is satisfied for \( \lambda_1 \), since \( \lambda_1 \) has multiplicity 1
• for $\lambda_2$, since

$$\lambda_2 I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

has rank 2, we have $n - \text{rank}(\lambda_2 I - A) = 3 - 2 = 1$, which is not the multiplicity of $\lambda_2$

• this means that we cannot find two independent eigenvectors corresponding to $\lambda_2$

• thus $A$ is not diagonalizable (in fact, $A$ is in Jordan canonical form)
Example

show that the following statement is true: if $A \in \mathbb{C}^{n \times n}$ has distinct eigenvalues, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$, then $A$ is diagonalizable (Lecture 11-22).

Solution. the characteristic polynomial of $A$ has order of $n$. since its roots are distinct, we have $n$ different eigenvalues with multiplicity of 1. without loss of generality, let’s show $\dim(\mathcal{N}(\lambda_1 I - A)) = 1$

• clearly, $v_i \neq \alpha v_1$ for $i = 2, \ldots, n$

• since $v_1 \notin \text{span}\{v_2, \ldots, v_n\}$, $\{v_1, \ldots, v_n\}$ is linearly independent

• also $v_i \notin \mathcal{N}(\lambda_1 I - A)$ for $i = 2, \ldots, n$ which implies $\dim(\mathcal{N}(\lambda_1 I - A)) = 1$

• therefore, $\dim(\mathcal{N}(\lambda_i I - A)) = 1$ for $i = 1, \ldots, n$
Example

show that the following statement is true: It is possible for $A$ to have repeated eigenvalues, but still be diagonalizable (Lecture 11-22)

Solution.

• as an example, assume

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

with $\chi(\lambda) = (\lambda - 1)^2(\lambda - 2)$ has multiplicity of 2 for $\lambda_1 = 1$ but still is diagonalizable

• here, $e_1$ and $e_2$ are eigenvectors for $\lambda_1$, and $e_3$ is for $\lambda_2$, so the condition still holds
Eigen decomposition theorem

- if a matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable, $A$ can be written as an eigen decomposition

$$A = T \Lambda T^{-1} = \sum_{i=1}^{n} \lambda_i v_i w_i^T$$

where $T = [v_1 \cdots v_n]$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and $v_i$ is the eigenvector corresponding to eigenvalue $\lambda_i$

- if $A$ is symmetric, i.e., $A = A^T$, then $A$ is diagonalizable and can be written as

$$A = Q \Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$
where \( Q = [q_1 \cdots q_n] \) is an orthogonal matrix

- you may see that this is useful for positive definiteness soon!
Application: Modal form

suppose $A \in \mathbb{C}^{n \times n}$ is diagonalizable, i.e., $A = T \Lambda T^{-1}$

then we can rewrite $\dot{x} = Ax$ as $\dot{\tilde{x}} = \Lambda \tilde{x}$ where $\tilde{x} = T^{-1}x$
Application: Real modal form

when eigenvalues (hence $T$) are complex, system can be put in real modal form

• assume $A \in \mathbb{R}^{n \times n}$ is diagonalizable, i.e., $A = T \Lambda T^{-1}$, and has a complex eigenvalue $\lambda_i = \sigma_i + j\omega_i$ with corresponding eigenvector $v_i$, i.e., $Av_i = \lambda_i v_i$

• taking the conjugate, we have $A \bar{v}_i = \bar{\lambda}_i \bar{v}_i$, which implies $\bar{v}_i$ is also an eigenvector corresponding to the eigenvalue $\bar{\lambda}_i$

• we can select $\lambda_{i+1} = \bar{\lambda}_i = \sigma_i - j\omega_i$

from this argument, we can show:

$$Av_i = \lambda_i v_i$$

$$= (\sigma_i + j\omega_i)(\Re(v_i) + j\Im(v_i))$$
\[
\sigma_i \Re(v_i) - \omega_i \Im(v_i) + j(\omega_i \Re(v_i) + \sigma_i \Im(v_i))
\]

and

\[
Av_{i+1} = A\bar{v}_i = \bar{\lambda}_i \bar{v}_i = (\sigma_i - j\omega_i)(\Re(v_i) - j\Im(v_i)) = \sigma_i \Re(v_i) - \omega_i \Im(v_i) - j(\omega_i \Re(v_i) + \sigma_i \Im(v_i))
\]

and thus,

\[
A\Re(v_i) = \sigma_i \Re(v_i) - \omega_i \Im(v_i)
\]

\[
A\Im(v_i) = \omega_i \Re(v_i) + \sigma_i \Im(v_i)
\]
which can be rewritten in a matrix form:

\[
\begin{bmatrix}
\Re(v_i) & \Im(v_i)
\end{bmatrix}
\begin{bmatrix}
\sigma_i & \omega_i \\
-\omega_i & \sigma_i
\end{bmatrix}
= A \begin{bmatrix}
\Re(v_i) & \Im(v_i)
\end{bmatrix}
\]

now, let’s bundle the real and complex eigenvalues together. assume \(\lambda_i \in \mathbb{R}\) for \(i = 1, \ldots, r\) and \(\lambda_i \in \mathbb{C} \setminus \mathbb{R}\) for \(i = r + 1, \ldots, n\) and let

\[
\Lambda_r = [\lambda_1 \cdots \lambda_r], \quad M_i = \begin{bmatrix}
\sigma_i & \omega_i \\
-\omega_i & \sigma_i
\end{bmatrix},
\]

and

\[
S = \begin{bmatrix}
v_1 & \cdots & v_r & \Re(v_{r+1}) & \Im(v_{r+1}) & \Re(v_{r+3}) & \Im(v_{r+3}) & \cdots
\end{bmatrix}
\]

since \(S\) is invertible, we finally have

\[
S^{-1}AS = \text{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1})
\]
Example: Standard form for LDS

• given LDS as $\dot{x} = Ax + Bu$ and $y = Cx$

• suppose $A$ is diagonalizable, i.e., $A = T\Lambda T^{-1}$

• let’s change coordinates with $x = T\tilde{x}$

then,

$$
\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}AT\tilde{x} + T^{-1}Bu = \Lambda\tilde{x} + \tilde{B}u
$$

$$
y = CT\tilde{x} = \tilde{C}\tilde{x}
$$

where $\tilde{B} = T^{-1}B$ and $\tilde{C} = CT$
Example

• suppose that $A \in \mathbb{R}_{+}^{n \times n}$ is diagonalizable, and its eigenvalues are nonnegative.

• consider a discrete-time linear dynamic system, $x(t + 1) = Ax(t)$, and a given nonnegative initial state, i.e., $x_i(0) \geq 0$ for $i = 1, \ldots, n$

• how can we find the asymptotic growth rate of sum, i.e., $\frac{1^T x(t+1)}{1^T x(t)}$ as $t \to \infty$?

Solution.

• since $A$ is diagonalizable, we can rewrite $A = \sum_{i=1}^{n} \lambda_i v_i w_i^T$ as above
• without loss of generality let's assume that \( \lambda_1 \geq \ldots \geq \lambda_n \geq 0 \), and let's take \( m \) as the index of the largest eigenvalue for which \( x(0) \) is not in the associated nullspace, i.e., \( w_m^T x(0) \neq 0 \), and \( w_i^T x(0) = 0 \) for \( i = 1, \ldots, m - 1 \).

• then, \( 1^T x(t) = 1^T A^t x(0) = \sum_{i=1}^{n} \lambda_i^t (1^T v_i)(w_i^T x(0)) \)

• as \( t \to \infty \), we get \( 1^T x(t) \approx \lambda_m^t (1^T v_m)(w_m^T x(0)) \)

• therefore, \( \frac{1^T x(t+1)}{1^T x(t)} \to \lambda_m \) as \( t \to \infty \)
Transfer matrix and impulse matrix

• assume LDS is given as \( \dot{x}(t) = Ax(t) + Bu(t) \) and \( y(t) = Cx(t) + Du(t) \)

• taking Laplace transform, we can get:

\[
\begin{align*}
    sX(s) - x(0) &= AX(s) + BU(s) \\
    \Leftrightarrow \quad X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \\
    \Rightarrow \quad Y(s) &= C(sI - A)^{-1}x(0) + \underbrace{(C(sI - A)^{-1}B + D)}_{H(s)}U(s)
\end{align*}
\]

• since \( (sI - A)^{-1} \to \mathcal{L}e^{tA} \), we can get:

\[
\begin{align*}
    x(t) &= e^{tA}x(0) + e^{tA}Bu(t) \quad \text{and} \quad y(t) = Ce^{tA}x(0) + \underbrace{(Ce^{tA}B + D\delta(t))}_{h(t)}u(t)
\end{align*}
\]

where \( H(s) \) and \( h(t) \) are transfer matrix and impulse matrix, respectively
• discretization: if we sample the states and outputs with interval $h$ as $x_d(k) = x(kh)$ and $y_d(k) = y(kh)$, we can get:

$$x_d(k+1) = x(kh + h)$$
$$= e^{(kh+h)A}x(0) + e^{tA}Bu(t)|_{t=kh+h}$$
$$= e^{hA} \left( e^{khA}x(0) + e^{tA}Bu(t)|_{t=kh} \right)$$
$$+ \int_{t=0}^{h} e^{(h-\tau)A}Bu(\tau + kh) d\tau$$

$$= e^{hA}x_d(k) + \int_{t=0}^{h} e^{(h-\tau)A}Bu(\tau + kh) d\tau$$

and

$$y_d(k) = Ce^{khA}x(0) + h(t) * u(t)|_{t=kh}$$
$$= C x_d(k) + Du(kh)$$
Example

when inputs are piecewise constant, *i.e.*, \( u(t) = u_d(k) \) for \( kh \leq t < (k+1)h \), rewrite the linear dynamic system in the discretized form

*Solution*. with the piecewise constant inputs, the above system of equations can be rewritten as:

\[
\begin{align*}
xd(k+1) &= e^{hA}xd(k) + \int_{t=0}^{h} e^{(h-\tau)A} Bu(\tau + kh) d\tau \\
&= \left( e^{hA} \right)_{A_d} x_d(k) + \left( \int_{t=0}^{h} e^{\tau A} d\tau \right)_{B_d} B u_d(k)
\end{align*}
\]

and

\[
\begin{align*}
y_d(k) &= C x_d(k) + D u_d(k)
\end{align*}
\]