Basic concepts from linear algebra

- nullspace
- range
- rank and conservation of dimension
Prerequisites

We assume that you are familiar with the basic definitions of the following concepts from lecture 3:

• vector spaces

• subspaces

• independence

• span

• basis

• dimension
Nullspace of a matrix

• For a matrix \( A \in \mathbb{R}^{m \times n} \), the nullspace is defined as,

\[
\mathcal{N}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.
\]

• Is \( \mathcal{N}(A) \subseteq \mathbb{R}^N \) a vector subspace of \( \mathbb{R}^n \)? Can you prove it?

\textit{Solution:} take two vectors \( v_1, v_2 \in \mathcal{N}(A) \), and scalars \( \alpha_1, \alpha_2 \in \mathbb{R} \). Then,

\[
A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 Av_1 + \alpha_2 Av_2 = 0.
\]

So \( \alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{N}(A) \).

• Roughly speaking, to verify that a set \( \mathcal{V} \subseteq \mathbb{R}^n \) is a subspace, we need only check that it is closed under vector addition and scalar multiplication.
Example 1

Let

\[ P = \begin{bmatrix} A \\ A + B \\ A + B + C \end{bmatrix}. \]

- True or false?
  \[ N(P) = N(A) \cap N(B) \cap N(C) \]

- Note that \( N(P) \) is a set, and \( N(A) \cap N(B) \cap N(C) \) is also a set.

- We say that two sets \( X \) and \( Y \) are equal if \( z \in X \Rightarrow z \in Y \), and \( z \in Y \Rightarrow z \in X \).

Solution:

- syntax check: Two subspaces can only be equal if they contain vectors of the same size:
LHS: if $A \in \mathbb{R}^{m \times n}$, then $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{m \times n}$. Therefore, $P$ has $n$ columns, and so $\mathcal{N}(P)$ is a subspace of $\mathbb{R}^n$.

RHS: $\mathcal{N}(A)$, $\mathcal{N}(B)$, and $\mathcal{N}(C)$ are all subspaces of $\mathbb{R}^n$, and hence their intersection must also be a subspace of $\mathbb{R}^n$.

- show that $x \in \mathcal{N}(P) \Rightarrow \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$.

Let $x \in \mathcal{N}(P)$. This implies

\[
\begin{align*}
Ax &= 0 \\
Ax + Bx &= 0 \\
Ax + Bx + Cx &= 0
\end{align*}
\]

\[
\begin{align*}
x \in \mathcal{N}(A) & \Rightarrow x \in \mathcal{N}(B) & x \in \mathcal{N}(A) \cap \mathcal{N}(B) \\
x \in \mathcal{N}(A) \cap \mathcal{N}(B) & \Rightarrow x \in \mathcal{N}(C) & x \in \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C).
\end{align*}
\]

- show that $x \in \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C) \Rightarrow x \in \mathcal{N}(P)$

This is trivial: if $Ax = 0$, $Bx = 0$, $Cx = 0$ then $Ax + Bx = 0$ and $Ax + Bx + Cx = 0$, so $x \in \mathcal{N}(P)$. 
Example 2

Is this true or false?

\[ \mathcal{N}(A^T A) = \mathcal{N}(A). \]

Solution:

• syntax check: If \( A \in \mathbb{R}^{m \times n} \) then \( A^T A \in \mathbb{R}^{n \times n} \), so \( \mathcal{N}(A^T A) \) and \( \mathcal{N}(A) \) are both subspaces of \( \mathbb{R}^n \).

• show that \( x \in \mathcal{N}(A) \Rightarrow x \in \mathcal{N}(A^T A) \):

\[
Ax = 0 \Rightarrow (A^T A)x = A^T(Ax) = A^T 0 = 0,
\]

so \( x \in \mathcal{N}(A^T A) \).
• show that $x \in \mathcal{N}(A^T A) \Rightarrow x \in \mathcal{N}(A)$:

Suppose $x \in \mathcal{N}(A^T A)$. Then, $A^T A x = 0$ and so

$$x^T A^T A x = (A x)^T (A x) = \|A x\|^2 = 0 \Rightarrow \|A x\| = 0.$$

– norm of a vector is zero if and only if the vector is equal to zero, i.e., $\|z\| = 0 \Leftrightarrow z = 0$.
– $\|A x\| = 0$ therefore implies that $A x = 0$, and so $x \in \mathcal{N}(A)$.  

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Range of a Matrix

• The range of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as,

$$\mathcal{R}(A) = \{ Ax \mid x \in \mathbb{R}^n \}$$

• $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$ (check this for yourself!).

• Roughly speaking, it is the set of vectors in $\mathbb{R}^m$ that can be ‘hit’ by the linear mapping $y = Ax$.

• Given a set of linear equations $y = Ax$ (where $y$ and $A$ are known), a solution $x$ exists if and only if $y$ can be hit by the linear mapping $y = Ax$, i.e., $y \in \mathcal{R}(A)$. 
Two more useful subspaces

- $\mathcal{R}(A^T)$: the space spanned by the rows of $A^T$; also called the row-space.
- $\mathcal{N}(A^T)$: also called the left nullspace.
- $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$ are sometimes called the four fundamental subspaces,
- By the end of lecture 4, you will understand the relationships between these subspaces and their properties.
Example 3

Draw $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A^T)$ for the matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \quad (1)$$

Solution:

- $\mathcal{R}(A)$: note that $A$ has only one independent column: $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.
  - So $\mathcal{R}(A)$ is all the scalar multiples of $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

- $\mathcal{R}(A^T)$: $A^T$ also has only one independent column, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.
  - So $\mathcal{R}(A^T)$ is all the scalar multiples of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

- Using a similar argument for $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$ gives us the diagrams shown in figure 1. (Do this yourself!)
• Vectors in $\mathcal{R}(A)$ are orthogonal to vectors in $\mathcal{N}(A^T)$; vectors in $\mathcal{R}(A^T)$ are orthogonal to vectors in $\mathcal{N}(A)$.

• This is no coincidence, and in fact, in lecture 4, you’ll see why this is true.
Rank and conservation of dimension

The rank of a matrix $A \in \mathbb{R}^{m \times n}$ defined as

$$\text{rank}(A) = \dim \mathcal{R}(A).$$

Some facts:

- **rank**$(A) = \text{rank}(A^T)$.

- **rank**$(A)$ is the maximum number of independent columns (or rows) of $A$, hence **rank**$(A) \leq \min(m, n)$.

- **rank**$(A) + \dim \mathcal{N}(A) = n$. (Conservation of dimension)
  
  - **rank**$(A)$ is the dimension of the set ‘hit’ by the mapping $y = Ax$,
  - $\mathcal{N}(A)$ is the dimension of the set of $x$ mapped to zero by $y = Ax$.
  - $n$ is the number of degrees of freedom in $x$.
  - so, roughly speaking, each degree of freedom in the input is either mapped to zero, or ends up in the output.
Example 4

• What is \( \text{rank}(A) \), the matrix labeled (1), from the previous example? 
  \textit{answer: } 1.

• Given \( A \in \mathbb{R}^{m \times n} \), what is

  – \( \dim \mathcal{R}(A^T) + \dim \mathcal{N}(A) \)? \textit{answer: } \( n \).
  – \( \dim \mathcal{R}(A) + \dim \mathcal{N}(A^T) \)? \textit{answer: } \( m \).
Full rank matrices

\[ A \in \mathbb{R}^{m \times n} \text{ is full rank if } \text{rank}(A) = \min(m, n). \]

- For square matrices, full rank means the matrix is nonsingular and invertible.

- For skinny matrices \((m \geq n)\), full rank means that the columns of \(A\) are independent, and sometimes we say that \(A\) is full column rank.

- For fat matrices \((m \leq n)\), full rank means that the rows of \(A\) are independent, sometimes referred to as full row rank.
Example 5

Suppose $A \in \mathbb{R}^{m \times n}$ is fat ($m < n$) and full rank. Show that $A$ has a right inverse, i.e., there is a matrix $B \in \mathbb{R}^{n \times m}$ such that $AB = I$. Is the right inverse unique?

**Solution:**

- If $A$ is fat and full rank, then $\text{rank}(A) = m$, which means that $\dim \mathcal{R}(A) = m$, and so $\mathcal{R}(A) = \mathbb{R}^m$ (i.e., $A$ is an onto matrix).

- Therefore, the equation $Ax = y$ can be solved for any $y \in \mathbb{R}^m$.

- So we can find $[b_1, \ldots, b_m]$ that satisfy
  
  $$Ab_i = e_i, \quad i = 1, \ldots, m.$$ 

- Form $B = [b_1, \ldots, b_m]$. Then,
  
  $$AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix} = I,$$
so $B$ is a right inverse of $A$.

- **uniqueness:**
  
  - By the conservation of dimension

  $$\dim \mathcal{N}(A) = n - \dim \mathcal{R}(A) = n - m \geq 1.$$ 

  - Let’s take a nonzero vector $z \in \mathcal{N}(A)$, and any nonzero vector $w \in \mathbb{R}^m$, then for any right inverse $B$,

    $$A(B + zw^T) = AB + Azw^T = I + 0 = I,$$

    so $B + zw^T$ is also a right inverse.
  
  - Since $B$ and $B + zw^T$ are different, we have two different right inverses. So the right inverse is not unique.
Example 6

Suppose $A \in \mathbb{R}^{m \times n}$ is skinny ($m > n$) and full rank. Show that $A$ has a left inverse, i.e., there is a matrix $B \in \mathbb{R}^{n \times m}$ such that $BA = I$.

**Solution.**

- If $A$ is skinny and full rank, then $\text{rank}(A) = n$, which means that $A$ has $n$ independent rows and columns.

- Thus $A^T$ is fat and full rank, and so the equation $A^T x = y$ can be solved for any $y \in \mathbb{R}^n$.

- Similar to the previous example, we can find a matrix $C \in \mathbb{R}^{m \times n}$ such that $A^T C = I$.

- Let $B = C^T$. $B$ is a left inverse of $A$. 