Lecture 10
Solution via Laplace transform and matrix exponential

- Laplace transform
- solving $\dot{x} = Ax$ via Laplace transform
- state transition matrix
- matrix exponential
- qualitative behavior and stability
Laplace transform of matrix valued function

suppose \( z : \mathbb{R}_+ \rightarrow \mathbb{R}^{p \times q} \)

Laplace transform: \( Z = \mathcal{L}(z) \), where \( Z : D \subseteq \mathbb{C} \rightarrow \mathbb{C}^{p \times q} \) is defined by

\[
Z(s) = \int_0^\infty e^{-st} z(t) \, dt
\]

• integral of matrix is done term-by-term

• convention: upper case denotes Laplace transform

• \( D \) is the domain or region of convergence of \( Z \)

• \( D \) includes at least \( \{ s \mid \Re s > a \} \), where \( a \) satisfies \( |z_{ij}(t)| \leq \alpha e^{at} \) for \( t \geq 0, \ i = 1, \ldots, p, \ j = 1, \ldots, q \)
Derivative property

\[ \mathcal{L}(\dot{z}) = sZ(s) - z(0) \]

to derive, integrate by parts:

\[
\mathcal{L}(\dot{z})(s) = \int_{0}^{\infty} e^{-st} \dot{z}(t) \, dt
\]

\[
= e^{-st} z(t) \bigg|_{t=0}^{t=\infty} + s \int_{0}^{\infty} e^{-st} z(t) \, dt
\]

\[
= sZ(s) - z(0)
\]
Laplace transform solution of $\dot{x} = Ax$

consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^n$

- take Laplace transform: $sX(s) - x(0) = AX(s)$

- rewrite as $(sI - A)X(s) = x(0)$

- hence $X(s) = (sI - A)^{-1}x(0)$

- take inverse transform

$$x(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) x(0)$$
Resolvent and state transition matrix

- $(sI - A)^{-1}$ is called the *resolvent* of $A$

- resolvent defined for $s \in \mathbb{C}$ except eigenvalues of $A$, i.e., $s$ such that $\det(sI - A) = 0$

- $\Phi(t) = L^{-1} \left((sI - A)^{-1}\right)$ is called the *state-transition matrix*; it maps the initial state to the state at time $t$:

  $$x(t) = \Phi(t)x(0)$$

  (in particular, state $x(t)$ is a linear function of initial state $x(0)$)
Example 1: Harmonic oscillator

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x \]

Solution via Laplace transform and matrix exponential
\[ sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -\frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \]

(eigenvalues are \( \pm i \))

state transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -\frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \]

a rotation matrix (\(-t\) radians)

so we have \( x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \)
Example 2: Double integrator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x
\]
\[ sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \]

(eigenvalues are 0, 0)

state transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

so we have \( x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \)
Characteristic polynomial

\[ \mathcal{X}(s) = \det(sI - A) \] is called the *characteristic polynomial* of \( A \)

- \( \mathcal{X}(s) \) is a polynomial of degree \( n \), with leading (i.e., \( s^n \)) coefficient one
- roots of \( \mathcal{X} \) are the eigenvalues of \( A \)
- \( \mathcal{X} \) has real coefficients, so eigenvalues are either real or occur in conjugate pairs
- there are \( n \) eigenvalues (if we count multiplicity as roots of \( \mathcal{X} \))
Eigenvalues of $A$ and poles of resolvent

$i, j$ entry of resolvent can be expressed via Cramer’s rule as

\[
(-1)^{i+j} \frac{\det \Delta_{ij}}{\det(sI - A)}
\]

where $\Delta_{ij}$ is $sI - A$ with $j$th row and $i$th column deleted

- $\det \Delta_{ij}$ is a polynomial of degree less than $n$, so $i, j$ entry of resolvent has form $f_{ij}(s)/\mathcal{X}(s)$ where $f_{ij}$ is polynomial with degree less than $n$

- poles of entries of resolvent must be eigenvalues of $A$

- but not all eigenvalues of $A$ show up as poles of each entry (when there are cancellations between $\det \Delta_{ij}$ and $\mathcal{X}(s)$)
Matrix exponential

\[(I - C)^{-1} = I + C + C^2 + C^3 + \cdots \text{ (if series converges)}\]

- series expansion of resolvent:

\[(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots\]

(valid for \(|s|\) large enough) so

\[\Phi(t) = \mathcal{L}^{-1} ((sI - A)^{-1}) = I + tA + \frac{(tA)^2}{2!} + \cdots\]
• looks like ordinary power series

\[ e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots \]

with square matrices instead of scalars . . .

• define **matrix exponential** as

\[ e^M = I + M + \frac{M^2}{2!} + \cdots \]

for \( M \in \mathbb{R}^{n \times n} \) (which in fact converges for all \( M \))

• with this definition, state-transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) = e^{tA} \]
Matrix exponential solution of autonomous LDS

solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$
• matrix exponential is *meant* to look like scalar exponential

• some things you’d guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold

• but *many things you’d guess are wrong*

**example:** you might guess that $e^{A+B} = e^A e^B$, but it’s false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$
however, we do have $e^{A+B} = e^A e^B$ if $AB = BA$, i.e., $A$ and $B$ commute

thus for $t, s \in \mathbb{R}$, $e^{(tA+sA)} = e^{tA} e^{sA}$

with $s = -t$ we get

$$e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I$$

so $e^{tA}$ is nonsingular, with inverse

$$(e^{tA})^{-1} = e^{-tA}$$
**example:** let's find $e^A$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

we already found

$$e^{tA} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so, plugging in $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let's check power series:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

since $A^2 = A^3 = \cdots = 0$
Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

**interpretation:** the matrix $e^{tA}$ propagates initial condition into state at time $t$

more generally we have, for any $t$ and $\tau$,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

**interpretation:** the matrix $e^{tA}$ propagates state $t$ seconds forward in time (backward if $t < 0$)
• recall first order (forward Euler) *approximate* state update, for small $t$:

\[
x(\tau + t) \approx x(\tau) + t \dot{x}(\tau) = (I + tA)x(\tau)
\]

• *exact* solution is

\[
x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau)
\]

• forward Euler is just first two terms in series
Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample $x$ at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k + 1) = e^{(t_{k+1}-t_k)A}z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k + 1) = e^{hA}z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)
Piecewise constant system

consider time-varying LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots \\ A_i & t_i \leq t < t_{i+1} \end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \ldots e^{(t_{3}-t_2)A_2} e^{(t_{2}-t_1)A_1} e^{t_1A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)

Solution via Laplace transform and matrix exponential
Qualitative behavior of $x(t)$

suppose $\dot{x} = Ax, \ x(t) \in \mathbb{R}^n$

then $x(t) = e^{tA}x(0)\ ; \ X(s) = (sI - A)^{-1}x(0)$

$i$th component $X_i(s)$ has form

$$X_i(s) = \frac{a_i(s)}{X(s)}$$

where $a_i$ is a polynomial of degree $< n$

thus the poles of $X_i$ are all eigenvalues of $A$ (but not necessarily the other way around)
first assume eigenvalues $\lambda_i$ are distinct, so $X_i(s)$ cannot have repeated poles

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

where $\beta_{ij}$ depend on $x(0)$ (linearly)

eigenvalues determine (possible) qualitative behavior of $x$:

- eigenvalues give exponents that can occur in exponentials
- real eigenvalue $\lambda$ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution
- complex eigenvalue $\lambda = \sigma + j\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t} \cos(\omega t + \phi)$ in solution
• \( \Re \lambda_j \) gives exponential growth rate (if \( > 0 \)), or exponential decay rate (if \( < 0 \)) of term

• \( \Im \lambda_j \) gives frequency of oscillatory term (if \( \neq 0 \))
now suppose $A$ has repeated eigenvalues, so $X_i$ can have repeated poles

express eigenvalues as $\lambda_1, \ldots, \lambda_r$ (distinct) with multiplicities $n_1, \ldots, n_r$, respectively ($n_1 + \cdots + n_r = n$)

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t)e^{\lambda_j t}$$

where $p_{ij}(t)$ is a polynomial of degree $< n_j$ (that depends linearly on $x(0)$)
Stability

we say system \( \dot{x} = Ax \) is \textit{stable} if \( e^{tA} \to 0 \) as \( t \to \infty \)

meaning:

- state \( x(t) \) converges to 0, as \( t \to \infty \), no matter what \( x(0) \) is

- all trajectories of \( \dot{x} = Ax \) converge to 0 as \( t \to \infty \)

\textbf{fact:} \( \dot{x} = Ax \) is stable if and only if all eigenvalues of \( A \) have negative real part:

\[ \Re \lambda_i < 0, \quad i = 1, \ldots, n \]
the ‘if’ part is clear since

$$\lim_{t\to\infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re \lambda < 0$

we’ll see the ‘only if’ part next lecture

more generally, $\max_i \Re \lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if $< 0$)