1 A method for rapidly driving the state to zero

Consider the discrete-time linear dynamical system

\[ x(t + 1) = Ax(t) + Bu(t), \]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times k}, k < n, \) and \( B \) is full rank. Our goal is to choose an input sequence \( u(t) \) that causes \( x(t) \) to converge to zero as \( t \to \infty \). An engineer proposes the following simple method: at time \( t \), choose \( u(t) \) in order to minimize \( \|x(t + 1)\| \). The engineer argues that this scheme will work well because the norm of the state is made as small as possible at every step.

(a) Find an explicit expression for the proposed input in terms of \( x(t), A, \) and \( B \).

(b) Suppose \( u(t) \) is chosen as suggested by the engineer. Show that there exists a matrix \( F \) such that \( x(t + 1) = Fx(t) \). Give an expression for \( F \) in terms of \( A \) and \( B \).

(c) Consider the specific instance of the problem with

\[ A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Compare the behavior of \( x(t + 1) = Ax(t) \) and \( x(t + 1) = Fx(t) \) for three different randomly chosen initial conditions. Determine which system is stable.