1 Positive-quadrant invariance
Consider a system $\dot{x} = Ax$, where $A \in \mathbb{R}^{2 \times 2}$. (The results of this problem can be generalized to higher dimensions, but you only need to focus on the specific case when the state has dimension $n = 2$.) We say that the system is positive-quadrant invariant (PQI) if we have that $x_1(t) \geq 0$ and $x_2(t) \geq 0$ for all $t \geq T$ whenever we have that $x_1(T) \geq 0$ and $x_2(T) \geq 0$. In other words, if the state starts inside (or enters) the positive quadrant (also called the first quadrant), then the state remains in the positive quadrant forever.

(a) Find the precise conditions on $A$ under which the system $\dot{x} = Ax$ is positive-quadrant invariant.

(b) Consider the following statement: if $\dot{x} = Ax$ is positive-quadrant invariant, then the eigenvalues of $A$ are real. Either prove that this statement is true, or give a counterexample showing that it is false.

2 Analyzing a continuous-time linear dynamical system
Consider the continuous-time linear dynamical system $\dot{x}(t) = Ax(t)$, where

$$A = \begin{bmatrix} -0.1005 & 1.0939 & 2.0428 & 4.4599 \\ -1.0880 & -0.1444 & 5.9859 & -3.0481 \\ -2.0510 & -5.9709 & -0.1387 & 1.9229 \\ -4.4575 & 3.0753 & -1.8847 & -0.1164 \end{bmatrix}.$$ 

(a) What are the eigenvalues of $A$? Is the system stable?

(b) Plot a few trajectories of the system. Is the qualitative behavior of the solutions consistent with the eigenvalues of the system?

(c) Find the matrix $Z$ such that $x(t + 15) = Zx(t)$.

(d) Find the matrix $Y$ such that $x(t - 20) = Yx(t)$.

(e) Comment on the relative magnitudes of the entries of $Y$ and $Z$.

(f) Find an $x(0)$ such that $x(10) = 1$.

3 Optimal preheating of an espresso cup
At time $t = 0$, boiling water (that is, water with temperature 100°C) is poured into an espresso cup; after $P$ seconds (the preheating time), the water is poured out, and espresso, with initial temperature 95°C, is poured in. (You can assume that these pouring operations occur instantaneously.) The espresso is then consumed exactly 15 s later (yes, instantaneously). The problem is to choose the preheating time $P$ in order to maximize the temperature of the espresso when it is consumed.
We can model this situation as follows. We take the temperature of the liquid in the cup (either water or espresso) as one state; we model the cup using an $n$-state finite-element model. The vector $x(t) \in \mathbb{R}^{n+1}$ gives the temperature distribution at time $t$: $x_1(t)$ is the temperature of the liquid (water or espresso) at time $t$, and $x_2(t), \ldots, x_{n+1}(t)$ are the temperatures of the elements in our model of the cup. All temperatures are in °C, and $t$ is in seconds. The dynamics are

$$\frac{d}{dt}(x(t) - 201) = A(x(t) - 201),$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is given. (The vector $201$ represents the ambient temperature.) The initial temperature distribution is

$$x(0) = \begin{bmatrix} 100 \\ 20 \\ \vdots \\ 20 \end{bmatrix}.$$

At $t = P$, the temperature of the liquid changes instantaneously from whatever value it had just before time $t = P$ to the temperature 95°C of the fresh espresso; the other states do not change instantaneously at time $t = P$. Note that the dynamics of the system are the same before and after preheating (because we assume that water and espresso behave in the same way thermally.)

The file `espresso_heating_data.m` defines the following variables.

- $A$, the dynamics matrix
- $n$, the number of states in the finite-element model of the cup
- $T_a$, the ambient temperature, 20°C
- $T_e$, the temperature of the espresso, 95°C
- $T_w$, the temperature of the water used for preheating, 100°C

Explain how to find the preheating time $P$ that maximizes the temperature of the espresso when it is consumed. Report the optimal value of $P$, and the corresponding temperature of the espresso when it is consumed. Report both quantities to an accuracy of one decimal place.