1 Certificates of infeasibility
Suppose I give you $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and I ask you to solve $Ax = b$. If the equation is consistent, then you can give me a solution $x \in \mathbb{R}^n$, and I can easily check that $Ax = b$. But if the equation is inconsistent, how can you convince me that there is no solution? Define the sets
\[ S_1 = \{ x \in \mathbb{R}^n \mid Ax = b \} \quad \text{and} \quad S_2 = \{ y \in \mathbb{R}^m \mid A^T y = 0, b^T y = 1 \}. \]

(a) Prove that if $S_2$ is nonempty, then $S_1$ must be empty. Thus, you can convince me that $Ax = b$ does not have a solution by giving me a vector $y \in \mathbb{R}^m$ such that $A^T y = 0$ and $b^T y = 1$. We call $y$ a certificate of infeasibility for $Ax = b$, and we call $S_1$ and $S_2$ weak alternatives.

(b) Prove that if $S_1$ is empty, then $S_2$ must be nonempty. Thus, if $Ax = b$ does not have a solution, then you can always find a certificate of infeasibility in $S_2$. Therefore, we call $S_1$ and $S_2$ strong alternatives.

(c) Suppose $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p \times n}$, and $d \in \mathbb{R}^p$ are given. The normal equations for the problem of minimizing $\|Ax - b\|$ subject to the constraint $Cx = d$ are
\[
\begin{bmatrix}
A^T A & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
A^T b \\
d
\end{bmatrix}.
\]
Prove that this system of equations has a solution whenever $Cx = d$ has a solution.

2 Meta-analysis with unknown measurements
Suppose $k$ different research papers report the results of different experiments designed to measure a common parameter $x \in \mathbb{R}^n$. The $i$th paper gives the estimate $\hat{x}_i \in \mathbb{R}^n$, which was computed by minimizing $\|A_i \hat{x}_i - y_i\|$, where $A_i \in \mathbb{R}^{m_i \times n}$ is the measurement matrix describing the $i$th experiment, and $y_i \in \mathbb{R}^{m_i}$ is the corresponding vector of measurements. You want to combine the results of these various experiments to obtain another estimate of $x$. You particular, you choose your estimate $\hat{x} \in \mathbb{R}^n$ in order to minimize
\[
\sum_{i=1}^k \|A_i \hat{x} - y_i\|^2.
\]
This type of analysis is called meta-analysis because you are analyzing the results of other experiments rather than running a new experiment. The catch in this problem is that paper $i$ reports the measurement matrix $A_i$, and the corresponding estimate $\hat{x}_i$, but not the vector $y_i$ of measurements.

(a) Explain how to use $A_1, \ldots, A_k$ and $\hat{x}_1, \ldots, \hat{x}_k$ to compute $\hat{x}$. State any assumptions that are needed for your method to work.
(b) Carry out your method on the data given in the file `meta_analysis_data.m`. This file defines cell arrays `A` and `xhat` whose `i`th entries are, respectively, the `i`th measurement matrix, $A_i$, and the `i`th parameter estimate, $\hat{x}_i$. Report your estimate $\hat{x}$.