1 Optimal preheating of an espresso cup

At time \( t = 0 \), boiling water (that is, water with temperature 100°C) is poured into an espresso cup; after \( P \) seconds (the preheating time), the water is poured out, and espresso, with initial temperature 95°C, is poured in. (You can assume that these pouring operations occur instantaneously.) The espresso is then consumed exactly 15 s later (yes, instantaneously). The problem is to choose the preheating time \( P \) in order to maximize the temperature of the espresso when it is consumed.

We can model this situation as follows. We take the temperature of the liquid in the cup (either water or espresso) as one state; we model the cup using an \( n \)-state finite-element model. The vector \( x(t) \in \mathbb{R}^{n+1} \) gives the temperature distribution at time \( t \): \( x_1(t) \) is the temperature of the liquid (water or espresso) at time \( t \), and \( x_2(t), \ldots, x_{n+1}(t) \) are the temperatures of the elements in our model of the cup. All temperatures are in °C, and \( t \) is in seconds. The dynamics are

\[
\frac{d}{dt}(x(t) - 201) = A(x(t) - 201),
\]

where \( A \in \mathbb{R}^{(n+1)\times(n+1)} \) is given. (The vector 201 represents the ambient temperature.) The initial temperature distribution is

\[
x(0) = \begin{bmatrix} 100 \\ 20 \\ \vdots \\ 20 \end{bmatrix}.
\]

At \( t = P \), the temperature of the liquid changes instantaneously from whatever value it had just before time \( t = P \) to the temperature 95°C of the fresh espresso; the other states do not change instantaneously at time \( t = P \). Note that the dynamics of the system are the same before and after preheating (because we assume that water and espresso behave in the same way thermally.)

The file \textit{espresso_heating_data.m} defines the following variables.

- \( A \), the dynamics matrix
- \( n \), the number of states in the finite-element model of the cup
- \( T_a \), the ambient temperature, 20°C
- \( T_e \), the temperature of the espresso, 95°C
- \( T_w \), the temperature of the water used for preheating, 100°C

Explain how to find the preheating time \( P \) that maximizes the temperature of the espresso when it is consumed. Report the optimal value of \( P \), and the corresponding temperature of the espresso when it is consumed. Report both quantities to an accuracy of one decimal place.
2 Analysis of diffusion in a compartment model

Suppose we are interested in the concentration of some substance as a function of time at different locations in a system. For example, we may be interested in the concentration of a drug as a function of time in different parts of the body. Compartment models are commonly used to model such a situation. We divide the system into \( n \) discrete units, called compartments, and we assume that the concentration of the substance of interest is constant throughout each compartment. Let \( x_i(t) \in \mathbb{R} \) be the concentration in compartment \( i \) at time \( t \), and let \( x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n \) denote the vector of concentrations at time \( t \). We assume that the concentration in compartment \( i \) evolves in time according to the equation

\[
x_i(t+1) = x_i(t) - \sum_{j \neq i} k_{ij} x_i(t) + \sum_{j \neq i} k_{ji} x_j(t), \quad i = 1, \ldots, n,
\]

where \( k_{ij} \) denotes the rate of diffusion from compartment \( i \) to compartment \( j \). (Note that \( k_{ij} \) and \( k_{ji} \) need not be equal.) The first summation gives the total diffusion out of compartment \( i \) at time \( t \), while the second summation gives the total diffusion into compartment \( i \) at time \( t \). We assume that the diffusion coefficients \( k_{ij} \) are known. We can represent such a model using a graph, as in figure 1.

![Figure 1](image)

**Figure 1** – a compartment model of the human body

Each of the nodes in the graph represents one of the compartments. If there is no edge from node \( i \) to node \( j \), then there is no diffusion directly from compartment \( i \) to compartment \( j \): that is, \( k_{ij} = 0 \). Each edge in the graph is labeled with the corresponding diffusion rate.

(a) Find a matrix \( A \in \mathbb{R}^{n \times n} \) such that

\[
x(t+1) = Ax(t).
\]
(b) For the rest of the problem, consider the specific compartment model with diffusion coefficients defined by the file `compartment_model_diffusion_data.m`. (The corresponding graph is the one given in figure 1.) Consider any initial condition $x(0)$ normalized such that $1^T x(0) = 1$. Compute the limit

$$\bar{x} = \lim_{t \to \infty} x(t),$$

which is the steady-state concentration profile. State any additional assumptions on $x(0)$ that are needed to justify your analysis.

(c) Let the difference between the current concentration profile $x(t)$, and the steady-state concentration profile $\bar{x}$ be

$$e(t) = x(t) - \bar{x}.$$ 

Compute the limit

$$r = \lim_{t \to \infty} \frac{\|e(t + 1)\|}{\|e(t)\|},$$

which gives the asymptotic rate at which $x(t)$ converges to its steady-state value. State any additional assumptions on $x(0)$ that are needed to justify your analysis.

(d) Simulate the system for $t = 0, \ldots, T = 500$ starting from the initial condition $x(0) = e_7$. (Intuitively, the substance is introduced into the body at time $t = 0$ via an intravenous injection into the left forearm.) Submit plots of $x_{10}(t)$ and $\|e(t)\|$ as functions of time. Use linear scales for $t$ and $x_{10}(t)$, and a logarithmic scale for $\|e(t)\|$. Briefly comment on your plots in relation to the theoretical results you derived above.

### 3 Convergence rate of power iteration

Consider the power iteration

$$x(t + 1) = \frac{1}{\|Ax(t)\|} Ax(t),$$

where $A \in \mathbb{R}^{n \times n}$ is given. We are interested in the limit

$$\bar{x} = \lim_{t \to \infty} x(t),$$

and how quickly $x(t)$ converges to this limiting value. Suppose $A$ has an eigenvalue decomposition:

$$\sum_{i=1}^{n} \lambda_i v_i w_i^T,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, $v_1, \ldots, v_n$ are unit right eigenvectors of $A$, and $w_1, \ldots, w_n$ are left eigenvectors of $A$. We assume that the eigenvalues are ordered such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

(a) Under certain assumptions, we have that

$$\bar{x} = \text{sgn}(w_1^T x(0)) v_1.$$ 

Prove this result, stating any assumptions that are needed for your analysis to hold.
(b) Define the convergence error

\[ e(t) = x(t) - \bar{x}. \]

Under certain assumptions, we have that

\[ \lim_{t \to \infty} \frac{\|e(t+1)\|}{\|e(t)\|} = \frac{|\lambda_2|}{|\lambda_1|}. \]

This gives the asymptotic rate of convergence of the iteration. Prove this result, stating any assumptions that are needed for your analysis to hold.

(c) The file `power_iteration_convergence_rate_data.m` defines the matrix \( A \), the initial vector \( x(0) \), and the time horizon \( T \). Compute \( x(t) \) for \( t = 1, \ldots, T \). Make a plot showing all of the components of \( x(t) \) as functions of time on a single set of axes. Also compute \( \bar{x} \), and indicate its components on your plot.

(d) Make a plot \( \|e(t)\| \) as a function of time; use a logarithmic scale for the vertical axis, and a linear scale for the horizontal axis. What is the asymptotic slope of this plot?

4 Maximizing \( y^T Ax \)

Suppose \( A \in \mathbb{R}^{m \times n} \), \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), and \( \|x\| = \|y\| = 1 \). What is the maximum possible value of \( y^T Ax \)? What values of \( x \) and \( y \) achieve this maximum value?

5 A search engine using low-rank approximation

In this problem we examine how linear algebra and low-rank approximation can be used to find matches to a search query in a set of documents. Consider a corpus with \( n \) documents, and a vocabulary of \( m \) terms. We define the term-by-document matrix \( A \in \mathbb{R}^{m \times n} \) such that

\[ a_{ij} = \text{the number of times term } i \text{ occurs in document } j. \]

A query is a vector \( q \in \mathbb{R}^m \) that expresses a criterion by which to select a document. We will consider query vectors that have 1s in the entries corresponding to the words that we want to search for, and 0s in all the other entries; however, more complex weighting schemes are possible. A simple measure of the relevance of document \( j \) to the query is given by

\[ a_j^T q, \]

where \( a_j \) denotes the \( j \)th column of \( A \). This criterion is biased towards large documents. For this reason, we use a normalized version of the inner product in order to rank the relevance of the documents:

\[ \frac{a_j^T q}{\|a_j\| \|q\|}. \]

Note that this normalized relevance criterion is the cosine of the angle between the document and query vectors. Since all the entries of the query and document vectors are nonnegative, our normalized relevance criterion lies between 0 and 1. Let \( \tilde{A} \) denote the normalized term-by-document matrix:

\[ \tilde{A} = \left[ \frac{1}{\|a_1\|} a_1 \ \cdots \ \frac{1}{\|a_n\|} a_n \right]. \]
If \( \tilde{q} = \frac{1}{\|q\|} q \) is a normalized query vector, then
\[
c = \tilde{A}^T \tilde{q}
\]
is a column vector giving an estimate of the relevance of each document to the query. The file `search_engine_data.m` defines the following variables.

- **term**, an \( m \times 1 \) cell array containing the search terms
- **document**, an \( n \times 1 \) cell array containing the URLs of the documents
- **A**, the \( m \times n \) term-by-document matrix

(a) Compute the normalized term-by-document matrix, \( \tilde{A} \). Make a stem plot of the singular values of \( \tilde{A} \).

(b) Perform a query for the word “students” \( (i = 53) \). What are the top 5 results?

(c) Let \( \hat{A}_r \) be the best rank-\( r \) approximation of \( \tilde{A} \). Compute \( \hat{A}_{32}, \hat{A}_{16}, \hat{A}_8, \) and \( \hat{A}_4 \). Perform a query for the word “students” using each of these matrices. Comment on the results.

(d) Are there advantages to using low-rank approximations rather than the full term-by-document matrix? You may assume that a very large number of searches will be performed before the term-by-document matrix is updated.

### 6 Voting patterns in the United States Senate

The file `senate_voting_patterns_data.m` contains the results of every roll-call vote held by the Senate of the 107th United States Congress (that is, in the years 2001–2002). In particular, running `senate_voting_patterns_data.m` defines the following variables.

- **senators**, a \( 102 \times 3 \) cell array (there are 102 rows because Senator Wellstone died during his term, and was replaced by Senator Barkley; and Senator Jeffords, who switched parties during his term, recorded votes as both a Republican and an Independent)
  - `senators{i,1}` is the last name of the \( i \)th Senator
  - `senators{i,2}` is the first name of the \( i \)th Senator
  - `senators{i,3}` is the party of the \( i \)th Senator
- **votes**, a \( 102 \times 633 \) matrix where \( \text{votes}(i,j) \) is the vote cast by Senator \( i \) in vote \( j \) (+1 if Senator \( i \) voted “Yea” on vote \( j \), −1 if Senator \( i \) voted “Nay” on vote \( j \), and 0 if Senator \( i \) did not participate in vote \( j \))

(a) Make a stem plot of the singular values of `votes`. How many significant singular values are there? The fraction of the variation in the voting data associated with the first two singular values is
\[
f_2 = \frac{\sigma_1^2 + \sigma_2^2}{\sum_{i=1}^r \sigma_i^2}.
\]

What is \( f_2 \) for the Senate voting data?
(b) Missing votes will complicate our subsequent analysis, so we use a low-rank model of the voting matrix to guess the missing votes. Initialize $A$ to the observed voting matrix; repeat the following steps until convergence:

- replace $A$ with its best rank-two approximation;
- replace the entries of $A$ corresponding to the known votes with their true values;
- replace the entries of $A$ corresponding to the unknown votes by setting nonnegative entries to $+1$, and negative entries to $-1$.

Report the numbers of $+1$s and $-1$s in your final matrix $A$.

(c) Let $A = UΣV^T$ be the singular-value decomposition of $A$. We want to find interpretations for the first two left singular vectors, $u_1$ and $u_2$.

(i) The function `spatial_plot`, which is available on the course website, can be used to generate color-coded scatter plots. Use this function to make a scatter plot of $(u_1)_i$ and $(u_2)_i$ where Republicans, Democrats, and Independents are represented by red, blue, and green markers, respectively. You can make such a plot using the command

```
spatial_plot(u1, u2, z, 3, eye(3));
```

where $u_1$ and $u_2$ are the first two left singular vectors of $A$, and $z$ is a vector where $z(i)$ is 0 if Senator $i$ is a Republican, 1 if Senator $i$ is an Independent, and 2 if Senator $i$ is a Democrat.

(ii) Make a scatter plot showing the fraction of votes in which each Senator voted with the majority of the other Senators. You can make such a plot using the command

```
spatial_plot(u1, u2, z, 10);
```

where $u_1$ and $u_2$ are the first two left singular vectors of $A$, and $z$ is a vector where $z(i)$ is the fraction of votes $j$ for which

$$A_{ij} = \text{sgn} \left( \sum_{i=1}^{102} A_{ij} \right).$$

(By default, `spatial_plot` uses the `cool` color map, which is pink for large values, and blue for small values.)

Based on these plots, give intuitive interpretations of $(u_1)_i$ and $(u_2)_i$.

(d) We can also find interpretations for the first two right singular vectors.

(i) Make a scatter plot of $(v_1)_j$ and $(v_2)_j$ where the colors indicate the total support received by vote $j$. In particular, the color should correspond to

$$z_j = \sum_{i=1}^{102} A_{ij}.$$
(ii) Make a scatter plot of \((v_1)_j\) and \((v_2)_j\) where the colors indicate the partisan support received by vote \(j\). In particular, let \(R \subseteq \{1, \ldots, 102\}\) be the set of Republicans, and let \(D \subseteq \{1, \ldots, 102\}\) be the set of Democrats. The color in your plot should correspond to

\[
    z_j = \frac{1}{|R|} \sum_{i \in R} A_{ij} - \frac{1}{|D|} \sum_{i \in D} A_{ij}.
\]

Based on these plots, give intuitive interpretations of \((v_1)_i\) and \((v_2)_i\).

7 Minimum-sensitivity estimation of the initial state

Consider a continuous-time autonomous linear dynamical system

\[
    \dot{x}(t) = Ax(t),
    \quad y(t) = Cx(t),
\]

where \(x(t) \in \mathbb{R}^n\) is the state, and \(y(t) \in \mathbb{R}\) is the output. At time \(T = 1\) you are required to give an estimate of \(x(0)\), the initial state of the system. In order to compute your estimate, you are allowed to choose times \(0 \leq t_1 < \cdots < t_n \leq T\) at which to measure the output of the system; you are only allowed to specify these times to decimal digits of accuracy. We have \(n\) measurements, and we need to compute the \(n\) components \(x_1(0), \ldots, x_n(0)\) of the initial state; this involves solving an \(n \times n\) system of linear equations. Let

\[
    y + \delta y = (y(t_1) + \delta y(t_1), \ldots, y(t_n) + \delta y(t_n))
\]

be the vector of noisy measurements that you are given, and let \(x(0) + \delta x(0)\) be the estimate of \(x(0)\) that you compute.

(a) Explain how to choose the measurement times \(t_1, \ldots, t_n\) in order to minimize the sensitivity:

\[
    \max_{x(0), \delta y \in \mathbb{R}^n} \left\{ \frac{\|\delta x(0)\|/\|x(0)\|}{\|\delta y\|/\|y\|} \right\}
\]

of your estimate of the initial state to the measurement errors.

(b) Apply your method to the data given in initial_state_sensitivity_data.m. Report your chosen measurement times and the corresponding sensitivity of your estimate to the measurement errors.