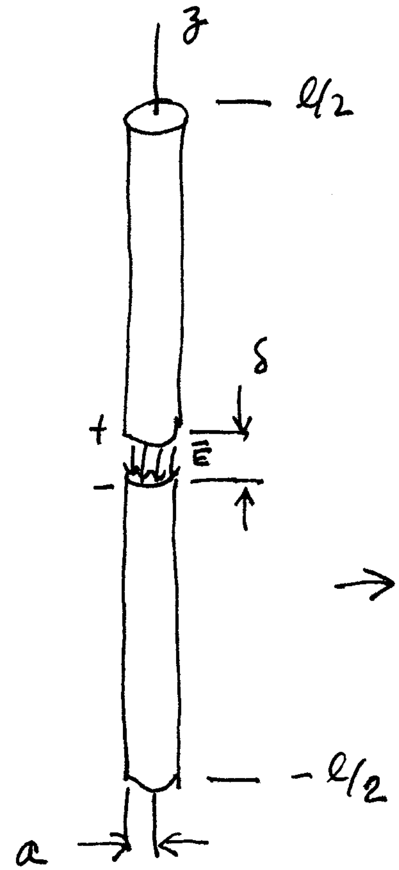
Hallen's Equation

We now want to combine the results for A_z from Pocklington's Equation with the radiation integral to find an integral equation for the dipole current. The new expression is called Hallen's Equation.

Recall:

Model of dipole for computation of input Z and current distribution



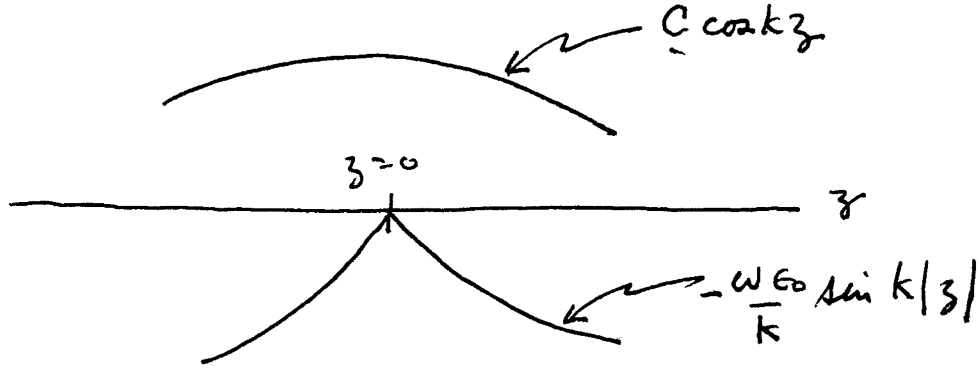
$$\left(\nabla_r \nabla_r \cdot + k^2\right) \int_{vol} \frac{\bar{J}(\bar{r}')\, e^{-jkR}}{R \cdot 4\pi}\, dv' = j\omega\epsilon_0 \bar{E}$$

$$\rightarrow \left(\frac{\partial^2}{\partial z^2} + k^2\right) \int_{surface} \frac{K_z(\phi', z')\, e^{-jkR}\, a\, d\phi'\, dz'}{4\pi R} = j\omega\epsilon_0 E_z$$

We attached $\left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z = j\omega\epsilon_0 \mu_0 \delta(z)$ (Dirac delta, here)

Solution for A_z is

$$A_z = C \cos kz - j \frac{\omega\epsilon_0}{k} \sin k|z|$$

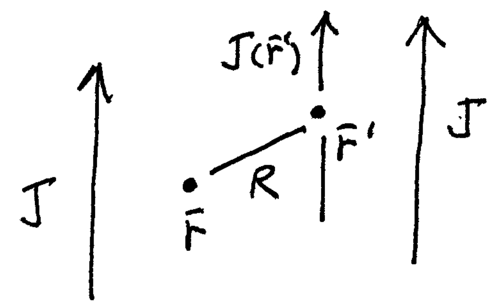


$$A_z(z) = \frac{1}{4\pi} \int_{\text{surface}} \frac{K_z(z')\, e^{-jkR}}{R}\, a\, d\phi'\, dz'$$

Regarding the singularity in the R ...

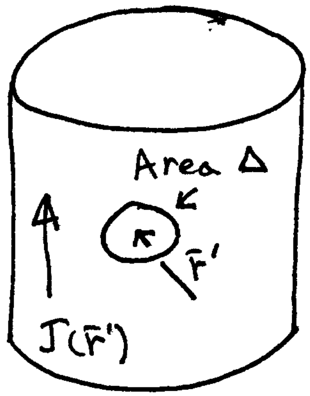
$$\int \frac{J(\bar{r}')e^{+jkR}}{4\pi R} ds'$$

$$R(\bar{r},\bar{r}') = |\bar{r}-\bar{r}'|$$

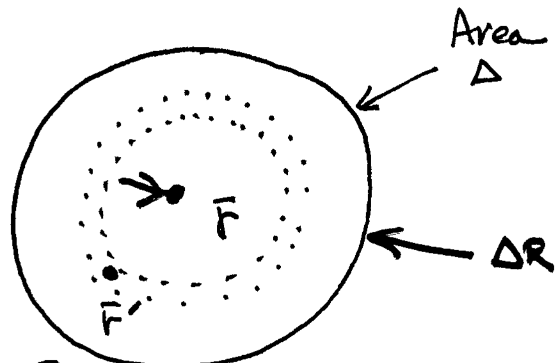


In Pocklington's Eq. $\bar{r}$ an $\bar{r}'$ both exist on the surface of a conductor (unlike our earlier problem of radiation at a distance from the source). Consequently there is a singularity in the integrand at $\bar{r}' = \bar{r}$. As the physical situation dictates, this singularity is integrable.

Consider $J(\bar{r}')$ on some surface



$$\int_{\text{surface}} = \int_{\text{surface}-\Delta} + \int_{\Delta}$$



$$\int_{\Delta} \frac{\bar{J}(\bar{r}')}{4\pi R} e^{+jkR} \, ds'$$

$$\underset{\Delta \to 0}{\overset{\text{MVThm}}{\approx}} \frac{\bar{J}(\bar{r})}{4\pi} \int_{\Delta} \frac{ds'}{R} = \frac{\bar{J}(\bar{r})}{4\pi} \int_{0}^{\Delta R} \frac{2\pi R}{R} \, dR = \frac{J(\bar{r})}{2} \cdot \Delta R \left\{ \begin{array}{l} \text{which} \\ \text{is} \\ \text{finite} \end{array} \right.$$

/... <u>Continuing</u> (our discussion of driving pt Z)

So,

$$\bar{R} = (\bar{r} - \bar{r}')$$
$$R = |\bar{r} - \bar{r}'|$$

$$\underbrace{\int_{-\ell/2}^{+\ell/2} \int_0^{2\pi} \frac{K_z(a, z') e^{-jkR}}{4\pi R} a\, d\phi' dz'}_{= A_z(z)} = C \cos kz - j \frac{\omega \epsilon_0}{2k} \sin k|z|$$

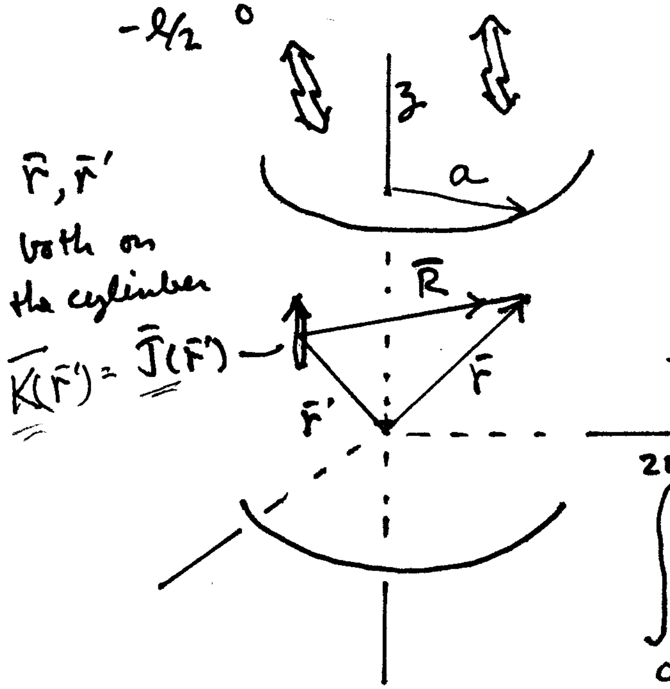
$= A_z(z)$ $\forall$ z on the dipole!

$$\boxed{\left.\int_{-\ell/2}^{\ell/2} \frac{I(z') e^{-jkR}}{4\pi R} dz' = C \cos kz - j \frac{\omega \epsilon_0}{2k} \sin k|z| \right\} \text{Hallen's Equation} \qquad R = \sqrt{(z-z')^2 + a^2}}$$

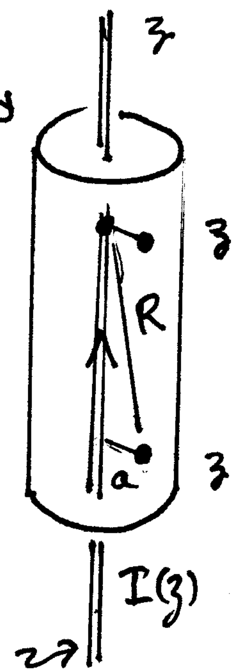
/...

What has happened here? The last approximation used is,

$$\int_{-l/2}^{l/2}\int_{0}^{2\pi} \frac{K_z(z')\, e^{-jkR}}{4\pi R}\, a\, d\phi'\, dz' \approx \int_{-l/2}^{l/2} \frac{I(z')\, e^{-jkR}}{4\pi R}\, dz'$$



$$R^2 = a^2 + (z-z')^2$$



$$\int_{0}^{2\pi} a\, K_z(\bar{r}')\, d\phi' = I(z'),$$

with $I(z')$ on axis

/...

/... Approximation (cont)

For typical cases, a/λ is small, and this approximation is good to about $1:10^{-3}$

Result of the above is to reduce wave equation in $\bar{A}, \bar{E}$, to a fairly simple F.H. equation of the first kind in $I(z)$.

There is still the unknown constant "C" - which we by passed.

$$I(z)\Big|_{z=\pm \ell/2} = 0 \Rightarrow 0$$

Regarding the Approximation

The approximation, above, is a fundamental one which greatly simplifies the computational complexity of the solutions. So we need to examine it in some detail.

Keeping in mind that

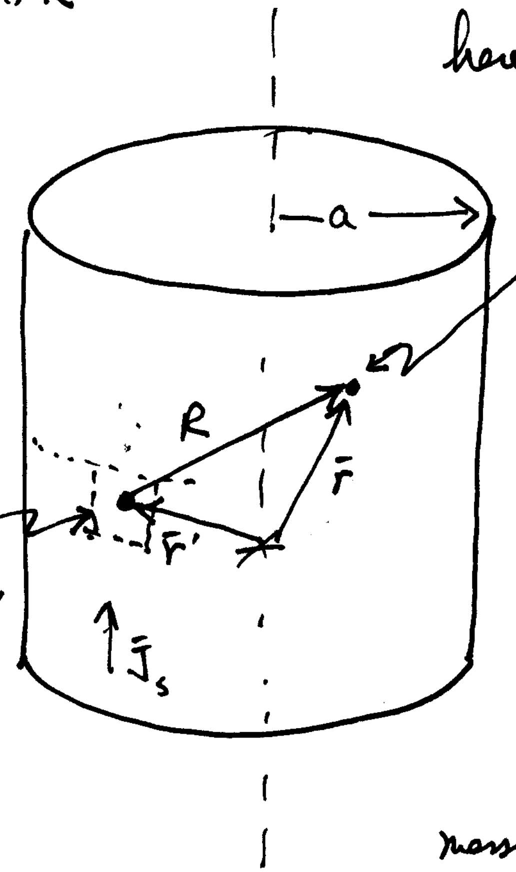
$$R = |\bar{r} - \bar{r}'|$$

/...

Approximation (cont.)

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi} \underline{K_z}\,\underline{(\phi, z')}\frac{e^{-jkR}}{4\pi R}\, a\, d\phi\, dz'$$

represents an integration over the surface of a cylinder here shown with exaggerated radius.



Observation point is at $\bar{r}$

$\bar{r} \to \bar{r}'$, $R \to 0$

Source point is at $d\bar{S}$ $= a\, d\phi\, dz'$ is located at r'

The point at $\bar{r}' = \bar{r}$ is an integrable singularity and presents no theoretical difficulties. But it is messy analytically and difficult numerically ...

/...

$$\int_{-\ell/2}^{\ell/2} \frac{I(z')e^{-jkR}}{4\pi R}\,dz', \quad R = \sqrt{a^2 + (z-z')^2}$$ is similar to the

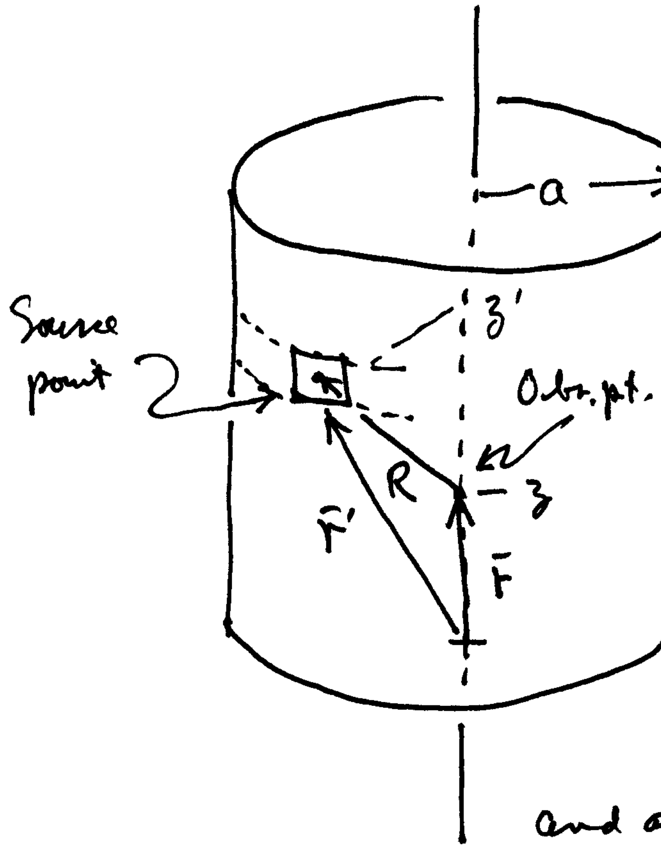
original expression, except that $\bar{r} \rightarrow z\,\bar{a}_z + 0\,\bar{a}_x + 0\,\bar{a}_y$ (the observation point is moved to the z-axis). In this case $\frac{e^{-jkR}}{4\pi R} \neq f(\phi)$, and the ϕ-integration can be carried out to give $I(z') = \int_0^{2\pi} K_z(\phi, z')\,a\,d\phi$, leading to the form at the top of the page.

/...

/...

The picture that goes with this last expression is given below.

$\bar{r}'$ is still located on the surface of the cylinder.



Physically, we have not moved $\bar{r}$ very far $(a/\lambda \ll 1)$, so $\bar{E}$ calculated at the new point would not be expected to change very much — an expectation borne out by careful analysis. At same time we have greatly reduced numerical and analytic complexity, and incorporated "a" into anal...

The subject of driving point impedance is discussed in Stutzman & Thiele, Chapter 7, pp 306 ff, where there is an alternative derivation of Pocklington's result.

Figure 7.2 of S&T is another way of indicating the physical approximation in shifting the current loading on the wire/dipole from the surface of the conductor to its axis. This figure is reproduced below.

Stutzman & Thiele

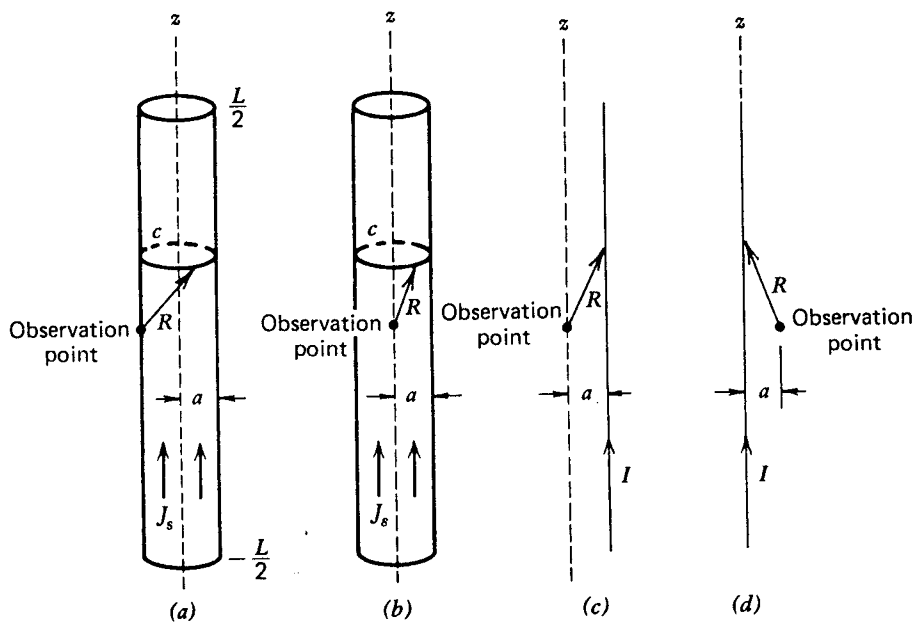


Figure 7-2 (*a*) Wire with surface current density J_s and observation point on the surface. (*b*) Wire with surface current density J_s and observation point on the wire axis. (*c*) Equivalent filamentary line source for the situation in (*b*). (*d*) Alternate representation of (*c*).