Midterm Exam

This is an in class 75 minute midterm.

You may not use any books, notes, or computer programs (e.g., Julia). Throughout this exam we use standard mathematical notation; and in particular, we do not use (and you may not use) notation from any computer language, or from any strange or non-standard mathematical dialect (e.g., physics).

Most of the questions are multiple choice. For these problems simply circle the appropriate response or responses. You do not need to give any justification for your answers to these questions. We will give partial credit for multiple choice problems left with no answer. If we can’t tell which response you are selecting, we will give zero credit.

For the last three problems you are asked for a free-form answer, which must be written between the lines below the problem.

All problems have equal weight. Some are easy. Others, not so much.

Name: ________________________________

SUID: ________________________________

(For EE103 staff only)

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1. **Vector equations.** For each of the following equations, circle the correct response.

(a) \[ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} . \]
   - This equation contains bad notation.
   - This equation is valid notation and is true.
   - This equation is valid notation but is false.

(b) \[ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = (1, 2, 1) . \]
   - This equation contains bad notation.
   - This equation is valid notation and is true.
   - This equation is valid notation but is false.

(c) \[ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = [1, 2, 1] . \]
   - This equation contains bad notation.
   - This equation is valid notation and is true.
   - This equation is valid notation but is false.

2. Let \( A \) and \( B \) be two \( m \times n \) matrices. Circle the correct response for each part.

(a) Suppose \( Ax = Bx \) holds for all \( n \)-vectors \( x \).
   - \( A = B \) always.
   - \( A = B \) sometimes.

(b) Suppose \( Ax = Bx \) for some nonzero \( n \)-vector \( x \).
   - \( A = B \) always.
   - \( A = B \) sometimes.

3. **Regression model.** Consider the regression model \( y = \beta^T x + v \), where \( y \) is the response, \( x \) is an 8-vector of features, \( \beta \) is an 8-vector of coefficients, and \( v \) is the offset term. For each of the following statements, circle the correct response.

(a) If \( \beta_3 > 0 \) and \( x_3 > 0 \), then \( y \geq 0 \).
   - True.
   - False.
(b) If $\beta_2 = 0$ then $y$ does not depend on $x_2$.
   - True.
   - False.

(c) If $\beta_6 = -0.8$, then increasing $x_6$ (keeping all other $x_i$'s the same) will decrease $y$.
   - True.
   - False.

(d) If $x$ and $\tilde{x}$ are feature vectors with corresponding responses $y$ and $\tilde{y}$, then $|y - \tilde{y}| \leq \|\beta\|\|x - \tilde{x}\|$.
   - True.
   - False.

4. **Linear independence under combination.** Suppose $S = \{a, b, c\}$ and $T = \{d, e, f\}$ are two linearly independent sets of $n$-vectors. For each of the sets given below, circle the correct statement.

(a) $\{a, b, c, d, e, f\}$
   - is always linearly independent.
   - is always linearly dependent.
   - could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

(b) $\{a + d, b + e, c + f\}$
   - is always linearly independent.
   - is always linearly dependent.
   - could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

(c) $\{a, a + b, a + b + c\}$
   - is always linearly independent.
   - is always linearly dependent.
   - could be linearly independent or linearly dependent, depending on the values of $a, \ldots, f$.

5. **Matrix sizes.** Suppose $A$, $B$, and $C$ are matrices that satisfy $A + BB^T = C$. Which of the following statements are necessarily true? Circle all that apply.
   - $A$ is square.
   - $A$ and $B$ have the same dimensions.
   - $A$, $B$, and $C$ have the same number of rows.
   - $B$ is a tall matrix.
6. True or false. For each statement, circle the correct answer.

(a) For any square matrix $A$, $(A + I)^3 = A^3 + 3A^2 + 3A + I$.
   - True.
   - False.

(b) If $n$-vectors $x$ and $y$ make an acute angle, then $\|x + y\| \geq \max\{\|x\|, \|y\|\}$.
   - True.
   - False.

(c) For any vector $a$, $\text{avg}(a) \leq \text{rms}(a)$.
   - True.
   - False.

7. Dynamics of an economy. Let $x_1, x_2, \ldots$ be $n$-vectors that give the level of economic activity of a country in years $1, 2, \ldots$, in $n$ different sectors (like energy, defense, manufacturing). Specifically, $(x_t)_i$ is the level of economic activity in economic sector $i$ (say, in billions of dollars) in year $t$. A common model that connects these economic activity vectors is $x_{t+1} = Ax_t$, where the $n \times n$ matrix $A$ is called the input-output matrix for the economy.

Which expression below gives the total economic activity across all sectors in year $t = 11$? Circle the correct one.

- $1^T(A^{10})x_1$.
- $1^T(A^{11})x_1$.
- $x_1^T(A^{11})1$.
- $1(A^{10})x_1$.

8. Linear independence. For each of the following matrices, circle the appropriate response.

(a) \[
\begin{bmatrix}
428 & 973 & -163 & 245 & -784 & 557 \\
352 & 869 & 0 & 781 & -128 & 120 \\
1047 & 45 & -471 & 349 & -721 & 781 \\
\end{bmatrix}
\]
   - The columns are linearly independent.
   - The columns are linearly dependent.
   - This is not an appropriate question for an in class 75 minute midterm.

(b) \[
\begin{bmatrix}
768 & 1121 & 3425 & 8023 \\
-2095 & -9284 & 5821 & -6342 \\
4093 & -3490 & -7249 & 8241 \\
834 & 1428 & 4392 & 5835 \\
-7383 & 1435 & 2345 & -293 \\
\end{bmatrix}
\]
• The columns are linearly independent.
• The columns are linearly dependent.
• This is not an appropriate question for an in class 75 minute midterm.

9. \textit{rms} and \textit{norm}. Let \(x\) and \(z\) be \(n\)-vectors, and \(y\) be an \(m\)-vector, with \(m \neq n\). For each of the following equations, circle the correct statement.

(a) \(\text{rms}((x, y)) = \text{rms}((\text{rms}(x), \text{rms}(y)))\)
   • always holds.
   • does not always hold.
   • makes no sense.

(b) \(\| (x, y) \| = \left\| \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \right\|\)
   • always holds.
   • does not always hold.
   • makes no sense.

(c) \(\text{rms}(x + z) \leq \text{rms}(x) + \text{rms}(z)\)
   • always holds.
   • does not always hold.
   • makes no sense.

10. \textit{Students, classes, and majors}. We consider \(m\) students, \(n\) classes, and \(p\) majors. Each student can be in any number of the classes (although we’d expect the number to range from 3 to 6), and can have any number of the majors (although the common values would be 0, 1, or 2). The data about the students’ classes and majors are given by an \(m \times n\) matrix \(C\) and an \(m \times p\) matrix \(M\), where

\[
C_{ij} = \begin{cases} 
1 & \text{student } i \text{ is in class } j \\
0 & \text{student } i \text{ is not in class } j 
\end{cases}
\]

and

\[
M_{ij} = \begin{cases} 
1 & \text{student } i \text{ is in major } j \\
0 & \text{student } i \text{ is not in major } j 
\end{cases}
\]

(a) Let \(E\) be the \(n\)-vector with \(E_i\) being the enrollment in class \(i\). Express \(E\) using matrix notation, in terms of the matrices \(C\) and \(M\), in the space below.
(b) Define the $n \times p$ matrix $S$ where $S_{ij}$ is the total number of students in class $i$ with major $j$. Express $S$ using matrix notation, in terms of the matrices $C$ and $M$, in the space below.

11. **Trimming a vector.** Find a matrix $A$ for which $Ax = (x_2, \ldots, x_{n-1})$, where $x$ is an $n$-vector. (Be sure to specify the size of $A$, and describe all its entries.)

Write your answer in the space below.

12. **Building a recommendation engine using $k$-means.** A set of $N$ users of a music-streaming app listens to songs from a library of $n$ songs over some period (say, a month). We describe this using an $N \times n$ matrix $P$ defined as

$$P_{ij} = \begin{cases} 1 & \text{user } i \text{ has played song } j \\ 0 & \text{user } i \text{ has not played song } j. \end{cases}$$

You can assume that if a user listens to a song, she likes it.

Your job (say, during a summer internship) is to design an algorithm that recommends to each user 10 songs that she has not listened to, but might like. (You can assume that for each user, there are at least 10 songs that she has not listened to.)

To do this, you start by running $k$-means on the columns of $P^T$. (It’s not relevant here, but a reasonable choice of $k$ might be 100 or so.) This gives the centroids $z_1, \ldots, z_k$, which are $n$-vectors.

Now what do you do? You can explain in words; you do not need to give a formula to explain how you make the recommendations for each user.

Write your answer in the space below.