Vectors

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Outline

Notation

Addition and scalar multiplication

Inner product

Complexity
Vectors

- A vector is an ordered list of numbers
- Written as
  \[
  \begin{pmatrix}
  -1.1 \\
  0.0 \\
  3.6 \\
  -7.2
  \end{pmatrix}
  \text{ or }
  \begin{pmatrix}
  -1.1 \\
  0.0 \\
  3.6 \\
  -7.2
  \end{pmatrix}
  \]
  or \((-1.1, 0, 3.6, -7.2)\)
- Numbers in a vector are called entries, coefficients, or elements.
- Length of vector is its size, length, or dimension.
- Vector above has dimension 4; its third entry is 3.6.
- Vector of length \(n\) is called an \(n\)-vector.
- Numbers are called scalars.
Vectors via symbols

- we’ll use symbols to denote vectors, e.g., $a$, $X$, $p$, $\beta$, $E^{\text{aut}}$
- other conventions: $g$, $\vec{a}$
- $i$th element of $n$-vector $a$ is denoted $a_i$
- if $a$ is vector above, $a_3 = 3.6$
- in $a_i$, $i$ is the index
- for an $n$-vector, indexes run from $i = 1$ to $i = n$
- warning: sometimes $a_i$ refers to the $i$th vector in a list of vectors
- two vectors $a$ and $b$ of the same size are equal if $a_i = b_i$ for all $i$
- we overload $=$ and write this as $a = b$
Block vectors

- suppose $b$, $c$, and $d$ are vectors with sizes $m$, $n$, $p$
- the *stacked vector* or *concatenation* (of $b$, $c$, and $d$) is
  \[ a = \begin{bmatrix} b \\ c \\ d \end{bmatrix} \]
- also called a *block vector*, with (block) entries $b$, $c$, $d$
- $a$ has size $m + n + p$
  \[ a = (b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_p) \]
Examples

- location or displacement in 2-D or 3-D
- color: \((R, G, B)\)
- quantities of \(n\) different commodities (or resources), e.g., a bill of materials
- portfolio: entries give shares (or $ value or fraction) held in each of \(n\) assets, with negative meaning short positions
- cash flow: \(x_i\) is payment in period \(i\) to us
- audio: \(x_i\) is the acoustic pressure at sample time \(i\) (sample times are spaced 1/44100 seconds apart)
- features: \(x_i\) is the value of \(i\)th feature or attribute of an entity
- word count: \(x_i\) is the number of times word \(i\) appears in a document
Zero, ones, and unit vectors

- $n$-vector with all entries 0 is denoted $0_n$ or just 0
- $n$-vector with all entries 1 is denoted $1_n$ or just 1
- a *unit vector* has one entry 1 and all others 0
- denoted $e_i$ where $i$ is entry that is 1
- unit vectors of length 3:

\[
e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
Sparsity

- a vector is *sparse* if many of its entries are 0
- can be stored and manipulated efficiently on a computer
- $\text{nnz}(x)$ is number of entries that are nonzero
- examples: zero vectors, unit vectors
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Vector addition

- $n$-vectors $a$ and $b$ and can be added, with sum denoted $a + b$
- to get sum, add corresponding entries:

\[
\begin{bmatrix}
0 \\
7 \\
3
\end{bmatrix}
+ \begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}
= \begin{bmatrix}
1 \\
9 \\
3
\end{bmatrix}
\]

- subtraction is similar
Properties of vector addition

- **Commutative**: $a + b = b + a$
- **Associative**: $(a + b) + c = a + (b + c)$
  (so we can write both as $a + b + c$)
- $a + 0 = 0 + a = a$
- $a - a = 0$

- these are easy and boring to verify
if 3-vectors $a$ and $b$ are displacements, $a + b$ is the sum displacement
Displacement from one point to another

- Displacement from point $q$ to point $p$ is $p - q$
Scalar-vector multiplication

- scalar $\alpha$ and $n$-vector $a$ can be multiplied
  $$\alpha a = (\alpha a_1, \ldots, \alpha a_n)$$
- also denoted $a\alpha$
- some properties:
  - associative: $(\beta \gamma)a = \beta(\gamma a)$
  - left distributive: $(\beta + \gamma)a = \beta a + \gamma a$
  - right distributive: $\beta(a + b) = \beta a + \beta b$
Linear combinations

- for vectors $a_1, \ldots, a_m$ and scalars $\beta_1, \ldots, \beta_m$, 

$$ \beta_1 a_1 + \cdots + \beta_m a_m $$

is a linear combination of the vectors
- $\beta_1, \ldots, \beta_m$ are the coefficients
- a very important concept
- examples:
  - audio mixing
  - replicating a cash flow
- a simple identity: for any $n$-vector $b$,

$$ b = b_1 e_1 + \cdots + b_n e_n $$
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Inner product

- *inner product* (or *dot product*) of \( n \)-vectors \( a \) and \( b \) is

\[
  a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
\]

- other notation used: \( \langle a, b \rangle, \langle a|b \rangle, (a, b), a \cdot b \)

- properties:
  - \( a^T b = b^T a \)
  - \( (\gamma a)^T b = \gamma (a^T b) \)
  - \( (a + b)^T c = a^T c + b^T c \)
Simple examples

- $e_i^T a = a_i$ (picks out $i$th entry)
- $1^T a = a_1 + \cdots + a_n$ (sum of entries)
- $a^T a = a_1^2 + \cdots + a_n^2$ (sum of squares of entries)
Examples

- $w$ is weight vector, $f$ is feature vector; $w^T f$ is score
- $p$ is vector of prices, $q$ is vector of quantities; $p^T q$ is total cost
- $c$ is cash flow, $d = (1, 1/(1 + r), \ldots, 1/(1 + r)^{n-1})$ is discount vector (with interest rate $r$); $d^T c$ is net present value (NPV) of cash flow
- $s$ gives portfolio holdings (in shares), $p$ gives asset prices; $p^T s$ is total portfolio value
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Complexity
Flop counts

- Computers store (real) numbers in *floating-point format*
- Basic arithmetic operations (addition, multiplication, . . . ) are called *floating point operations* or flops
- Complexity of an algorithm or operation: total number of flops needed, as function of the input dimension(s)
- This can be very grossly approximated
- Crude approximation of time to execute: computer speed/flops
- Current computers are around 1Gflop/sec (10^9 flops/sec)
- But this can vary by factor of 100
Complexity of vector addition, inner product

- $x + y$ needs $n$ additions, so: $n$ flops
- $x^T y$ needs $n$ multiplications, $n - 1$ additions so: $2n - 1$ flops
- we simplify this to $2n$ (or even $n$) flops for $x^T y$
- and much less when $x$ or $y$ is sparse