Matrix Examples

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many geometric transformations and mappings of 2-D and 3-D vectors can be represented via matrix multiplication $y = Ax$

for example, rotation by $\theta$:

$$y = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} x$$

(to get the entries, look at $Ae_1$ and $Ae_2$)
Selectors

- an $m \times n$ selector matrix: each row is a unit vector (transposed)

$$A = \begin{bmatrix}
  e^T_{k_1} \\
  \vdots \\
  e^T_{k_m}
\end{bmatrix}$$

- multiplying by $A$ selects entries of $x$:

$$Ax = (x_{k_1}, x_{k_2}, \ldots, x_{k_m}).$$

- examples: image cropping, down-sampling
Incidence matrix

- graph with $n$ vertices or nodes, $m$ (directed) edges or links
- incidence matrix is $n \times m$ matrix

$$A_{ij} = \begin{cases} 
1 & \text{edge } j \text{ points to node } i \\
-1 & \text{edge } j \text{ points from node } i \\
0 & \text{otherwise}
\end{cases}$$

- row $i$ associated with vertex $i$
- column $j$ associated with edge $j$; contains one entry $+1$ and one entry $-1$
Example

\[ A = \begin{bmatrix}
-1 & -1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 \\
0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \]
Flow conservation

- $m$-vector $x$ gives flows (of something) along the edges
- examples: heat, money, power, mass, people, ...
- $x_j > 0$ means flow follows edge direction
- $Ax$ is $n$-vector that gives the total or net flows
- $(Ax)_i$ is the net flow into node $i$
- $Ax = 0$ is flow conservation; $x$ is called a circulation
Convolution

- for $n$-vector $a$, $m$-vector $b$, their convolution is $c = a \ast b$,

$$c_k = \sum_{i+j=k+1} a_i b_j, \quad k = 1, \ldots, n + m - 1$$

- for example with $n = 4$, $m = 3$, we have

$$

c_1 = a_1 b_1 \\
c_2 = a_1 b_2 + a_2 b_1 \\
c_3 = a_1 b_3 + a_2 b_2 + a_3 b_1 \\
c_4 = a_2 b_3 + a_3 b_2 + a_4 b_1 \\
c_5 = a_3 b_3 + a_4 b_2 \\
c_6 = a_4 b_3
$$

- example: $(1, 0, -1) \ast (2, 1, -1) = (2, 1, -3, -1, 1)$
Polynomial multiplication

- $a$ and $b$ are coefficients of two polynomials,

$$p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, \quad q(x) = b_1 + b_2 x + \cdots + b_m x^{m-1}$$

- coefficients of product $p(x)q(x)$ are $c = a \ast b$:

$$p(x)q(x) = c_1 + c_2 x + \cdots + c_{n+m-1} x^{n+m-2}$$

- so $a \ast b = b \ast a$, $(a \ast b) \ast c = a \ast (b \ast c)$
function \( f(b) = a \ast b \) is linear; in fact \( c = T(b)a \) with

\[
T = \begin{bmatrix}
  b_1 & 0 & 0 & 0 \\
  b_2 & b_1 & 0 & 0 \\
  b_3 & b_2 & b_1 & 0 \\
  0 & b_3 & b_2 & b_1 \\
  0 & 0 & b_3 & b_2 \\
  0 & 0 & 0 & b_3
\end{bmatrix}
\]

\( T \) is a Toeplitz matrix (values on diagonals are equal)