Linear-Quadratic Estimation

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EE103
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November 30, 2016

Outline

Least squares state estimation

Example

Missing measurements

State estimation

- dynamic system model:

$$x_{t+1} = Ax_t + Bw_t, \quad y_t = Cx_t + v_t, \quad t = 1, 2, \ldots$$

- x_t is *state* (n-vector)
- y_t is *measurement* (p-vector)
- v_t is *measurement noise* or *measurement residual* (p-vector)
- w_t is *input* or *process noise* (m-vector)
- we know A, B, C, and measurements $y_1, \ldots, y_T$
- w_t, v_t are unknown, but assumed small
- *state estimation*: estimate/guess $x_1, \ldots, x_T$

Some common variations

- x_1 is known
- y_t not known for some t's ('missing measurements')
- matrices A, B, C are time-varying

Least squares state estimation

- choose estimates $\hat{x}_t$, $\hat{w}_t$, $\hat{v}_t$ by solving linearly constrained least squares problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{t=1}^{T} \|v_t\|_2^2 + \lambda \|w_t\|_2^2 \\
\text{subject to} & x_{t+1} = Ax_t + Bw_t, \quad t = 1, \ldots, T-1 \\
& y_t = Cx_t + v_t, \quad t = 1, \ldots, T-1
\end{array}
$$

- variables are $x_1, \ldots, x_T$, $w_1, \ldots, w_T$, $v_1, \ldots, v_T$
- $\lambda > 0$ is a parameter, trades off measurement and process errors

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Running example

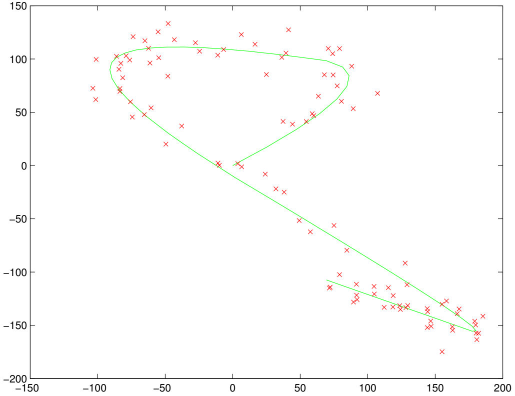
- we'll use a running example of a vehicle moving in 2-D
- $x_t = (p_t, z_t)$
  - 2-vector p_t is the 2-D position
  - 2-vector z_t is the 2-D velocity
- dynamics

$$p_{t+1} = p_t + z_t, \quad z_{t+1} = z_t + w_t$$

- so w_t is the force on the vehicle, which we generate
- $y_t = p_t + v_t$; we generate measurement noise v_t *randomly*
- $T = 100$
- we generate the data, so we know the true values $x_1^{\text{true}}, x_2^{\text{true}}, \ldots$
  (in real applications, of course, you don't know the true state)

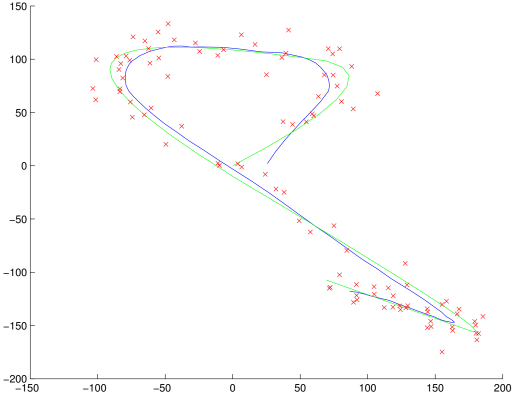
Measurements and true positions

- y_t shown as red crosses
- $y_t^{\text{true}} = Cx_t^{\text{true}}$ ('true' position) shown in green



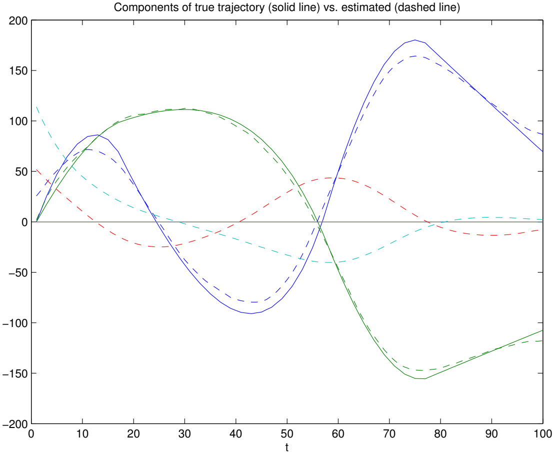
Least squares state estimation

- blue curve is $\hat{y}_t = C\hat{x}_t$, for $\lambda = 0.07$



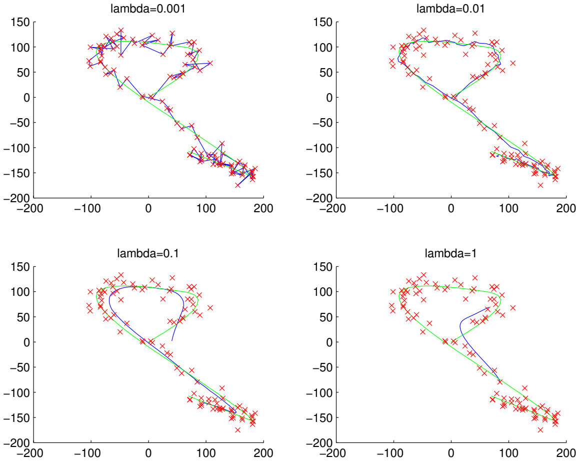
True and estimated state trajectories

- solid lines show true trajectory $(x_t^{\mathrm{true}})_i$
- dashed lines show estimates $(\hat{x}_t)_i$



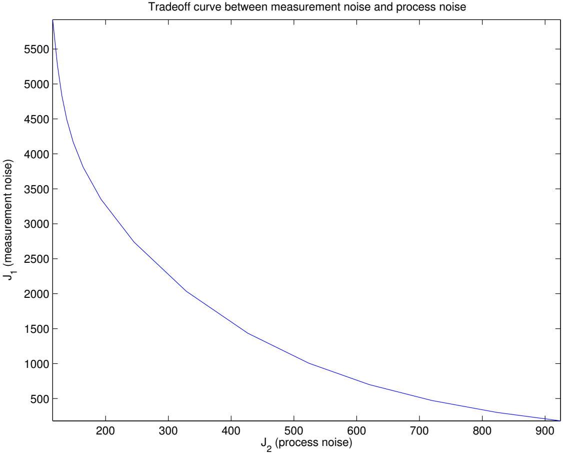
Varying λ

- state trajectory estimates using $\lambda = 1, 10, 100$



Varying λ

- objective is $J_1 + \lambda J_2$, $J_1 = \sum_{t=1}^{T} \|v_t\|_2^2$, $J_2 = \sum_{t=1}^{T} \|w_t\|_2^2$



Outline

Missing measurements

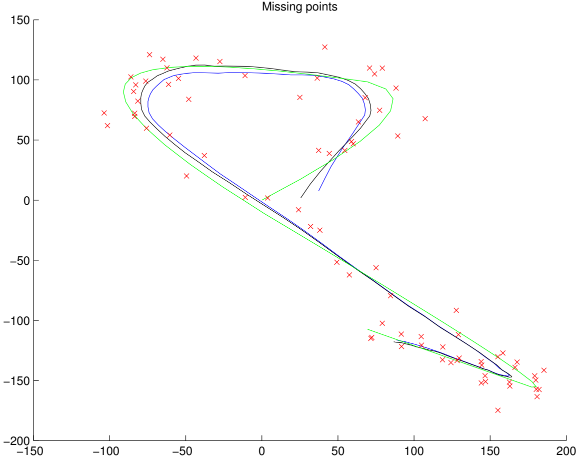
- suppose we only get measurements y_t for $t \in \mathcal{T} \subseteq \{1, \ldots, T-1\}$
- we form state estimate by solving problem

$$\begin{array}{ll} \text{minimize} & \sum_{t \in \mathcal{T}} \|v_t\|_2^2 + \lambda \sum_{t=1}^{T} \|w_t\|_2^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bw_t, \quad t = 1, \ldots, T-1 \\ & y_t = Cx_t + v_t, \quad t \in \mathcal{T} \end{array}$$

  with variables $x_1, \ldots, x_T$, $w_1, \ldots, w_T$, and v_t for $t \in \mathcal{T}$
- a linearly constrained least squares problem

Example

- same example, 20% of measurements removed
- blue curve shows $C\hat{x}_t$, $\lambda = 8$
- black curve shows estimate using all measurements

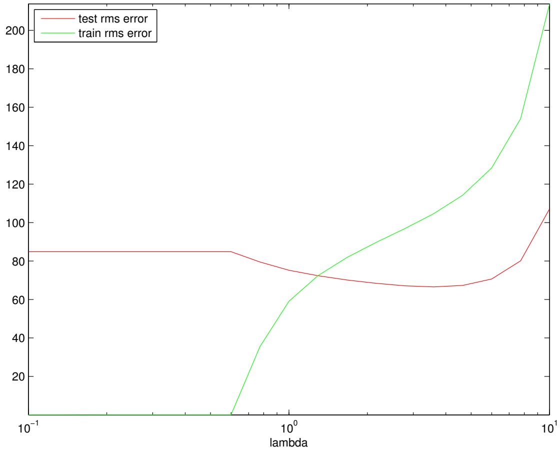


Cross validation

- randomly remove 20% (say) of the measurements and use as *test measurements*
- for many values of λ
  - carry out state estimation using other (*training*) measurements
  - evaluate RMS measurement residuals on test set
- choose λ to (approximately) minimize the RMS test residuals

Example

- training and test RMS errors versus λ
- confirms λ between 2 and 5 is good choice



Example

- result shown by $\lambda = 4.64$
- training y_t shown as red crosses, test y_t shown as red circles