Outline

Linear dynamical system

Control

Variations

Examples

Linear quadratic regulator
Linear dynamical system

\[ x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots \]

- \textit{n}-vector \( x_t \) is \textit{state} at time \( t \)
- \textit{m}-vector \( u_t \) is \textit{input} at time \( t \)
- \( n \times n \) matrix \( A \) is \textit{dynamics matrix}
- \( n \times m \) matrix \( B \) is \textit{input matrix}
- sequence \( x_1, x_2, \ldots \) is called \textit{state trajectory}
Simulation

- given $x_1, u_1, u_2, \ldots$ find $x_2, x_3, \ldots$
- can be done by recursion: for $t = 1, 2, \ldots$,

$$x_{t+1} = Ax_t + Bu_t$$
Vehicle example

consider a vehicle moving in a plane:

▶ sample position and velocity at times $\tau = 0, h, 2h, \ldots$
▶ 2-vectors $p_t$ and $v_t$ are position and velocity at time $ht$
▶ 2-vector $u_t$ gives applied force on the vehicle time $ht$
▶ friction force is $-\eta v_t$
▶ vehicle has mass $m$
▶ for small $h$,

$$m \frac{v_{t+1} - v_t}{h} \approx -\eta v_t + u_t,$$
$$\frac{p_{t+1} - p_t}{h} \approx v_t$$

▶ we use approximate state update

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t,$$
$$p_{t+1} = p_t + hv_t$$

Linear dynamical system
Vehicle state is 4-vector $x_t = (p_t, v_t)$

Dynamics recursion is

$$x_{t+1} = Ax_t + Bu_t,$$

where

$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & h/m \\ 0 & h/m \end{bmatrix}.$$
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- $x_1$ is given
- choose $u_1, u_2, \ldots, u_{T-1}$ to achieve some goals, e.g.,
  - terminal state should have some fixed value: $x_T = x^\text{des}$
  - $u_1, u_2, \ldots, u_{T-1}$ should be small, say measured as
    
    $$
    \|u_1\|^2 + \cdots + \|u_{T-1}\|^2
    $$

    (sometimes called ‘energy’)
- many control problems are linearly constrained least-squares problems
Minimum-energy state transfer

- given initial state $x_1$ and desired final state $x^{\text{des}}$
- choose $u_1, \ldots, u_{T-1}$ to minimize ‘energy’

\[
\begin{align*}
\text{minimize} & \quad ||u_1||^2 + \cdots + ||u_{T-1}||^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1 \\
& \quad x_T = x^{\text{des}}
\end{align*}
\]

variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$

- roughly speaking: find minimum energy inputs that steer the state to given target state over $T$ periods
State transfer example

vehicle model with $T = 100$, $x_1 = (10, 10, 10, -5)$, $x^{\text{des}} = 0$
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Output tracking

- \( y_t = C x_t \) is output (e.g., position)
- \( y_t \) should follow a desired trajectory, i.e., sum square tracking error
  \[
  \| y_2 - y_2^{\text{des}} \|^2 + \cdots + \| y_T - y_T^{\text{des}} \|^2
  \]
  should be small
- the output tracking problem is
  \[
  \text{minimize} \quad \sum_{t=2}^{T} \| y_t - y_t^{\text{des}} \|^2 + \rho \sum_{t=1}^{T-1} \| u_t \|^2 \\
  \text{subject to} \quad x_{t+1} = A x_t + B u_t, \quad t = 1, \ldots, T - 1 \\
  y_t = C x_t, \quad t = 1, \ldots, T - 1
  \]
  variables are \( x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, y_2, \ldots, y_T \)
- parameter \( \rho > 0 \) trades off control ‘energy’ and tracking error
Output tracking example

vehicle model with $T = 100$, $\rho = 0.1$, $x_1 = 0$, $y_t = p_t$ (position tracking)
Variations
Waypoints

- using output, can specify *waypoints*
- specify output (position) $w^{(k)}$ at time $t_k$ at $K$ total places

\[
\begin{align*}
\text{minimize} & \quad \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1 \\
& \quad Cx_{t_k} = w^{(k)}, \quad k = 1, \ldots, K
\end{align*}
\]

variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$
Waypoints example

- vehicle model
- \( T = 100, \ x_1 = (10, 10, 20, 0), \ x^{\text{des}} = 0 \)
- \( K = 4, \ t_1 = 10, \ t_2 = 30, \ t_3 = 40, \ t_4 = 80 \)
- \( w^{(1)} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}, \ w^{(2)} = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix}, \ w^{(3)} = \begin{bmatrix} 4.5 \\ 1 \end{bmatrix}, \ w^{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)
Waypoints example
Variations
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Linear quadratic regulator
Rendezvous

- we control two vehicles with dynamics
  \[ x_{t+1} = Ax_t + Bu_t, \quad z_{t+1} = Az_t + Bv_t \]

- final relative state constraint \( x_T = z_T \)

- formulate as state transfer problem:

  minimize \( \sum_{t=1}^{T-1} (\|u_t\|^2 + \|v_t\|^2) \)
  subject to
  \[ x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1, \]
  \[ z_{t+1} = Az_t + Bv_t, \quad t = 1, \ldots, T - 1, \]
  \[ x_T = z_T \]

  variables are \( x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, z_2, \ldots, z_T, v_1, \ldots, v_{T-1} \)
$x_1 = (0, 0, 0, -5), \ z_1 = (10, 10, 5, 0)$
Formation

- generalize rendezvous example to several vehicles
- final position for each vehicle defined relative to others (e.g., relative to a 'leader')
- leader has a final velocity constraint
Formation example

Examples
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Examples

Linear quadratic regulator
minimize energy while driving state to the origin:

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=2}^{T} \|x_t\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\
\text{subject to} & \quad x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T - 1
\end{align*}
\]

variables are \(x_2, \ldots, x_T, u_1, \ldots, u_{T-1}\)

\[
\sum_{t=2}^{T} \|x_t\|^2 \text{ is (sum square) regulation}
\]
\[
x = 0 \text{ is some desired (equilibrium, target) state}
\]
\[
\text{parameter } \rho > 0 \text{ trades off regulation versus control ‘energy’}
\]
LQR problem is a linearly constrained least-squares problem:

\[
\begin{align*}
\text{minimize} & \quad \|Fz\|^2 \\
\text{subject to} & \quad Gz = d
\end{align*}
\]

- variable \( z \) is \((x_2, \ldots, x_T, u_1, \ldots, u_{T-1})\)
- \( F, G \) depend on \( A, B, \rho \); \( d \) depends (linearly) on \( x_1 \)
- solution is \( \hat{z} = Hd \) for some \( H \)
- optimal first input \( \hat{u}_1 \) is a linear function of \( x_1 \), i.e.,

\[
\hat{u}_1 = Kx_1
\]

for some \( m \times n \) matrix \( K \) (called **LQR gain matrix**)

- finding \( K \) involves taking correct ‘slice’ of inverse KKT matrix
- entries of \( K \) depend on horizon \( T \), and converge as \( T \) grows large
State feedback control

- find $K$ for LQR problem (with large $T$)
- for each $t$,
  - measure state $x_t$
  - implement control $u_t = K x_t$
- with $u_t = K x_t$ is called state feedback control policy
- combine with (‘open-loop dynamics’) $x_{t+1} = Ax_t + Bu_t$ to get closed-loop dynamics

\[ x_{t+1} = (A + BK)x_t \]

- we can simulate open- and closed-loop dynamics to compare
variables are (small) deviations from operating point or *trim conditions*;

state is \( x_t = (w_t, v_t, \theta_t, q_t) \):
  - \( w_t \): velocity of aircraft along body axis
  - \( v_t \): velocity of aircraft perpendicular to body axis (down is positive)
  - \( \theta_t \): angle between body axis and horizontal (up is positive)
  - \( q_t = \dot{\theta}_t \): angular velocity of aircraft (pitch rate)

input is \( u_t = (e_t, f_t) \):
  - \( e_t \): elevator angle \( (e_t > 0 \text{ is down}) \)
  - \( f_t \): thrust
Linearized dynamics

for 747, level flight, 40000 ft, 774 ft/sec, dynamics are

\[ x_{t+1} = Ax_t + Bu_t, \]

where

\[
A = \begin{bmatrix}
0.99 & 0.03 & -0.02 & -0.32 \\
0.01 & 0.47 & 4.7 & 0.00 \\
0.02 & -0.06 & 0.40 & -0.00 \\
0.01 & -0.04 & 0.72 & 0.99
\end{bmatrix}, \quad B = \begin{bmatrix}
0.01 & 0.99 \\
-3.44 & 1.66 \\
-0.83 & 0.44 \\
-0.47 & 0.25
\end{bmatrix}
\]

- units: ft, sec, crad (= 0.01 rad ≈ 0.57°)
- discretization is 1 sec

Linear quadratic regulator
LQR gain

gain matrix $K$ converged for $T \approx 30$
LQR for 747 model

- LQR gain matrix (for $T = 100$, $\rho = 100$) is:

$$K = \begin{bmatrix} -0.038 & 0.021 & 0.319 & -0.270 \\ -0.061 & -0.004 & -0.120 & 0.007 \end{bmatrix}$$

- e.g., $K_{14} = -0.27$ is gain from pitch rate ($x_t^4$) to elevator angle ($u_t^1$)
747 simulation

\[ u_t = 0 \ ('open loop') \]

Linear quadratic regulator
\[ u_t = K x_t \ ('closed loop') \]