

Ballistics

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Outline

Dynamics

Simulations

Targeting

Robust targeting

Position, velocity, and force

- ▶ a projectile moves in 2-dimensional space
(for simplicity; real ones move in 3-dimensional space)
- ▶ sample position and velocity at times $\tau = 0, h, 2h, \dots$
- ▶ 2-vector p_t is position at time $\tau = th$ for $t = 0, 1, \dots$
- ▶ 2-vector v_t is velocity at time $\tau = th$ for $t = 0, 1, \dots$
- ▶ 2-vector f_t is total force acting on projectile at time $\tau = th$
- ▶ 4-vector $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$ is projectile *state* at time $\tau = th$

Force model

$$f_t = mg - \eta(v_t - w)$$

- ▶ 2-vector $g = (0, -9.8)$ is gravity
- ▶ 2-vector w is wind velocity (assumed constant)
- ▶ $v_t - w$ is relative velocity of projectile through air
- ▶ $\eta \in \mathbf{R}$ is *drag coefficient*
- ▶ $\eta(v_t - w)$ is *drag force*
- ▶ m is projectile mass
- ▶ 'ballistic' means the projectile has no other force acting on it (e.g., thrust or propulsion)

Dynamics

- ▶ approximating velocity as constant over time interval
 $th \leq \tau \leq (t+1)h$,

$$p_{t+1} = p_t + hv_t$$

- ▶ approximating force as constant over the time interval,

$$\begin{aligned}v_{t+1} &= v_t + (h/m)f_t \\ &= (1 - h\eta/m)v_t + (hg + h\eta w/m)\end{aligned}$$

- ▶ more compactly: $x_{t+1} = Ax_t + b$, with

$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ hg_1 + h\eta w_1/m \\ hg_2 + h\eta w_2/m \end{bmatrix}$$

Propagating state through time

- ▶ to propagate forward T time steps

$$x_1 = Ax_0 + b$$

$$x_2 = A(Ax_0 + b) + b = A^2x_0 + Ab + b$$

$$\vdots$$

$$x_T = A^T x_0 + (A^{T-1} + \dots + A + I)b$$

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Simulation parameters

- ▶ let's look at some trajectories, with parameters

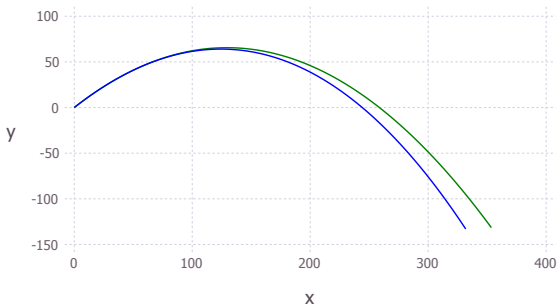
$$m = 5, \quad T = 100, \quad h = 0.1, \quad \eta = 0.05, \quad p_0 = 0$$

- ▶ we'll use various values of initial velocity v_0 , expressed in terms of
 - initial speed $\|v_0\|$
 - elevation $\theta = \tan^{-1}((v_0)_2/(v_0)_1)$
- ▶ we'll vary the wind velocity w too

Simulation: with and without wind

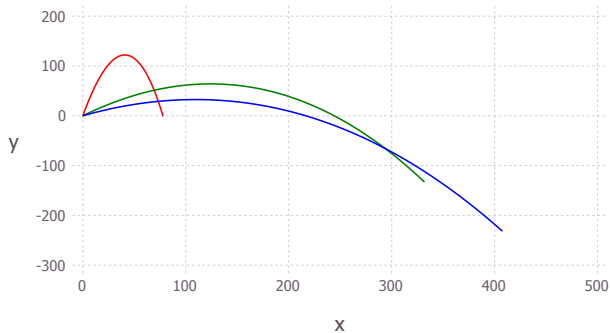
- ▶ initial speed $\|v_0\| = 50$, elevation $\theta = 45^\circ$
- ▶ no wind: $w = (0, 0)$
- ▶ with wind: $w = (-10, 0)$

(all future simulations include wind)



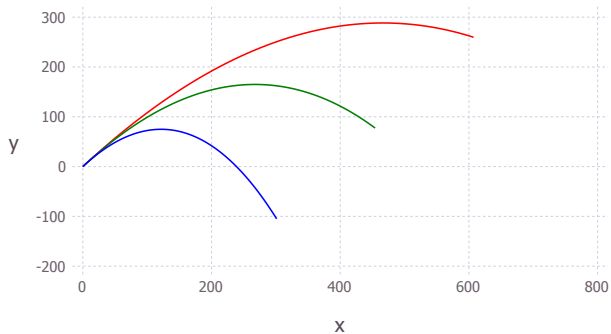
Simulation: varying elevation

- ▶ $\|v_0\| = 50$
- ▶ $\theta = 30^\circ, 45^\circ, 80^\circ$



Simulation: varying speed

- ▶ $\theta = 50$
- ▶ $\|v_0\| = 50, 75, 100$



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Targeting problem

- ▶ given
 - initial position p_0
 - parameters h, m, w, η
 - flight time Th
 - desired final position ('target') p_T
- ▶ find initial velocity v_0

- ▶ please note
 - this is not used for socially positive purposes
 - but it is one of the first historical applications

Final state

- ▶ final state is

$$\begin{aligned}x_T &= A^T x_0 + (A^{T-1} + \dots + A + I)b \\ &= Fx_0 + j\end{aligned}$$

where

$$F = A^T, \quad j = (A^{T-1} + \dots + A + I)b$$

- 4×4 matrix F maps initial state to final state
- 4-vector j is effect of gravity, wind on final state

Final position

- ▶ final position is

$$p_T = F_{11}p_0 + F_{12}v_0 + j_1$$

(F_{11} and F_{12} are 2×2 subblocks of F)

- ▶ write as $p_T = Cv_0 + d$, where $C = F_{12}$, $d = F_{11}p_0 + j_1$
- ▶ solving for v_0 we have (assuming $C = F_{12}$ is invertible)

$$v_0 = C^{-1}(p_T - d)$$

(note that C and d are known)

- ▶ gives formula for choosing v_0 (hence, $\|v_0\|$ and θ)

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- ▶ suppose we have uncertainty in the wind, drag coefficient, . . .
- ▶ uncertainty is modeled as K scenarios (particular values of parameters)
 - each scenario has its own $A^{(j)}, b^{(j)}$
 - hence its own $C^{(j)}, d^{(j)}$
- ▶ *robust targeting*: choose a single v_0 to minimize mean-square targeting error

$$\frac{1}{K} \sum_{j=1}^K \|C^{(j)}v_0 + d^{(j)} - p_T\|^2$$

Sample simulations

various masses, drag coefficients, and wind, with $T = 100$

