Lecture 5
Rational functions and partial fraction expansion

- (review of) polynomials
- rational functions
- pole-zero plots
- partial fraction expansion
- repeated poles
- nonproper rational functions

Polynomials and roots

polynomials

$$a(s) = a_0 + a_1 s + \cdots + a_n s^n$$

- a is a polynomial in the variable s
- a_i are the *coefficients* of a (usually real, but occasionally complex)
- n is the *degree* of a (assuming $a_n \neq 0$)

roots (or zeros) of a polynomial a: $\lambda \in \mathbf{C}$ that satisfy

$$a(\lambda) = 0$$

examples

- $a(s) = 3$ has no roots
- $a(s) = s^3 - 1$ has three roots: 1, $(-1 + j\sqrt{3})/2$, $(-1 - j\sqrt{3})/2$

factoring out roots of a polynomial

if a has a root at $s = \lambda$ we can *factor out* $s - \lambda$:

- dividing a by $s - \lambda$ yields a polynomial:

$$b(s) = \frac{a(s)}{s - \lambda}$$

  is a polynomial (of degree one less than the degree of a)

- we can express a as

$$a(s) = (s - \lambda)b(s)$$

  for some polynomial b

example: $s^3 - 1$ has a root at $s = 1$

$$s^3 - 1 = (s - 1)(s^2 + s + 1)$$

the **multiplicity** of a root λ is the number of factors $s - \lambda$ we can factor out, *i.e.*, the largest k such that

$$\frac{a(s)}{(s-\lambda)^k}$$

is a polynomial

example:

$$a(s) = s^3 + s^2 - s - 1$$

- a has a zero at $s = -1$
- $\dfrac{a(s)}{s+1} = \dfrac{s^3 + s^2 - s - 1}{s+1} = s^2 - 1$ also has a zero at $s = -1$
- $\dfrac{a(s)}{(s+1)^2} = \dfrac{s^3 + s^2 - s - 1}{(s+1)^2} = s - 1$ does not have a zero at $s = -1$

so the multiplicity of the zero at $s = -1$ is 2

Fundamental theorem of algebra

a polynomial of degree n has exactly n roots, counting multiplicities

this means we can write a in *factored form*

$$a(s) = a_n s^n + \cdots + a_0 = a_n(s - \lambda_1)\cdots(s - \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are the n roots of a

example: $s^3 + s^2 - s - 1 = (s+1)^2(s-1)$

the relation between the coefficients a_i and the λ_i is complicated in general, but

$$a_0 = a_n \prod_{i=1}^{n}(-\lambda_i), \quad a_{n-1} = -a_n \sum_{i=1}^{n} \lambda_i$$

are two identities that are worth remembering

Conjugate symmetry

if the coefficients $a_0, \ldots, a_n$ are real, and $\lambda \in \mathbf{C}$ is a root, *i.e.*,

$$a(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n = 0$$

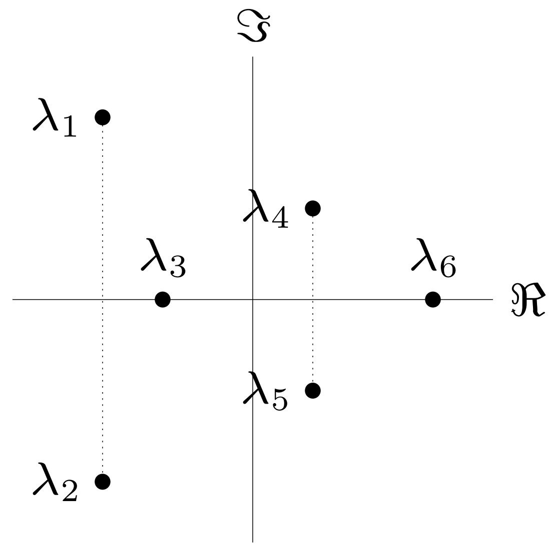
then we have

$$a(\overline{\lambda}) = a_0 + a_1\overline{\lambda} + \cdots + a_n\overline{\lambda}^n = \overline{(a_0 + a_1\lambda + \cdots + a_n\lambda^n)} = \overline{a(\lambda)} = 0$$

in other words: $\overline{\lambda}$ is also a root

- if λ is real this isn't interesting
- if λ is complex, it gives us another root for free
- complex roots come in *complex conjugate pairs*

example:



λ_3 and λ_6 are real; λ_1, λ_2 are a complex conjugate pair; λ_4, λ_5 are a complex conjugate pair

if a has real coefficients, we can factor it as

$$a(s) = a_n \left(\prod_{i=1}^{r} (s - \lambda_i) \right) \left(\prod_{i=r+1}^{m} (s - \lambda_i)(s - \overline{\lambda_i}) \right)$$

where $\lambda_1, \ldots, \lambda_r$ are the real roots; $\lambda_{r+1}, \overline{\lambda_{r+1}}, \ldots, \lambda_m, \overline{\lambda_m}$ are the complex roots

Real factored form

$$(s-\lambda)(s-\overline{\lambda}) = s^2 - 2(\Re\lambda)\, s + |\lambda|^2$$

is a quadratic with real coefficients

real factored form of a polynomial a:

$$a(s) = a_n \left(\prod_{i=1}^{r}(s-\lambda_i)\right)\left(\prod_{i=r+1}^{m}(s^2+\alpha_i s+\beta_i)\right)$$

- $\lambda_1, \ldots, \lambda_r$ are the real roots
- α_i, β_i are real and satisfy $\alpha_i^2 < 4\beta_i$

any polynomial with real coefficients can be factored into a product of

- degree one polynomials with real coefficients
- quadratic polynomials with real coefficients

example: $s^3 - 1$ has roots

$$s = 1, \quad s = \frac{-1 + j\sqrt{3}}{2}, \quad s = \frac{-1 - j\sqrt{3}}{2}$$

- complex factored form

$$s^3 - 1 = (s - 1)\left(s + (1 + j\sqrt{3})/2\right)\left(s + (1 - j\sqrt{3})/2\right)$$

- real factored form

$$s^3 - 1 = (s - 1)(s^2 + s + 1)$$

Rational functions

a *rational function* has the form

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n},$$

$i.e.$, a ratio of two polynomials (where a is not the zero polynomial)

- b is called the *numerator polynomial*
- a is called the *denominator polynomial*

examples of rational functions:

$$\frac{1}{s+1}, \quad s^2+3, \quad \frac{1}{s^2+1} + \frac{s}{2s+3} = \frac{s^3+3s+3}{2s^3+3s^2+2s+3}$$

rational function $F(s) = \dfrac{b(s)}{a(s)}$

polynomials b and a are not uniquely determined, $e.g.$,

$$\frac{1}{s+1} = \frac{3}{3s+3} = \frac{s^2+3}{(s+1)(s^2+3)}$$

(except at $s = \pm j\sqrt{3}\ldots$)

rational functions are closed under addition, subtraction, multiplication, division (except by the rational function 0)

Poles & zeros

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n},$$

assume b and a have no common factors (cancel them out if they do . . .)

- the m roots of b are called the *zeros* of F; λ is a zero of F if $F(\lambda) = 0$
- the n roots of a are called the *poles* of F; λ is a pole of F if $\lim_{s \to \lambda} |F(s)| = \infty$

the *multiplicity* of a zero (or pole) λ of F is the multiplicity of the root λ of b (or a)

example: $\dfrac{6s + 12}{s^2 + 2s + 1}$ has one zero at $s = -2$, two poles at $s = -1$

factored or *pole-zero* form of F:

$$F(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = k\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where

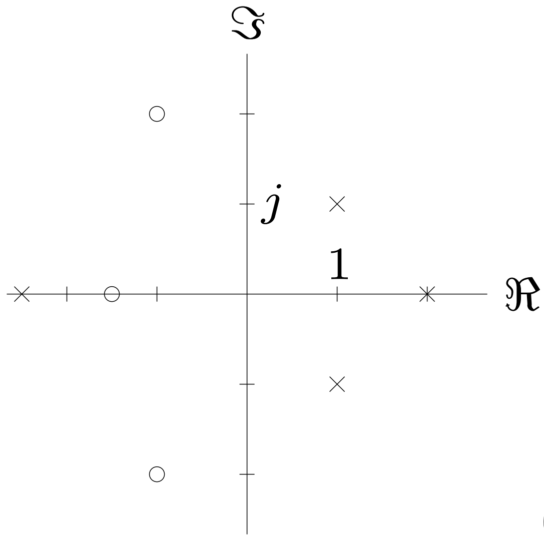
- $k = b_m/a_n$
- $z_1, \ldots, z_m$ are the zeros of F ($i.e.$, roots of b)
- $p_1, \ldots, p_n$ are the poles of F ($i.e.$, roots of a)

(assuming the coefficients of a and b are real) complex poles or zeros come in complex conjugate pairs

can also have *real factored form* . . .

Pole-zero plots

poles & zeros of a rational functions are often shown in a *pole-zero plot*



($\times$ denotes a pole; $\circ$ denotes a zero)

this example is for

$$
\begin{aligned}
F(s) &= k\frac{(s+1.5)(s+1+2j)(s+1-2j)}{(s+2.5)(s-2)(s-1-j)(s-1+j)} \\
&= k\frac{(s+1.5)(s^2+2s+5)}{(s+2.5)(s-2)(s^2-2s+2)}
\end{aligned}
$$

(the plot doesn't tell us k)

Partial fraction expansion

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

let's assume (for now)

- no poles are repeated, $i.e.$, all roots of a have multiplicity one
- $m < n$

then we can write F in the form

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

called **partial fraction expansion** of F

- $\lambda_1, \ldots, \lambda_n$ are the poles of F
- the numbers $r_1, \ldots, r_n$ are called the **residues**
- when $\lambda_k = \overline{\lambda_l}$, $r_k = \overline{r_l}$

example:

$$\frac{s^2-2}{s^3+3s^2+2s} = \frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

let's check:

$$\frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2} = \frac{-1(s+1)(s+2) + s(s+2) + s(s+1)}{s(s+1)(s+2)}$$

in partial fraction form, **inverse Laplace transform** is **easy:**

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}\right) \\ &= r_1 e^{\lambda_1 t} + \cdots + r_n e^{\lambda_n t} \end{aligned}$$

(this is real since whenever the poles are conjugates, the corresponding residues are also)

Finding the partial fraction expansion

two steps:

- find poles $\lambda_1, \ldots, \lambda_n$ (*i.e.*, factor $a(s)$)
- find residues $r_1, \ldots, r_n$ (several methods)

method 1: solve linear equations

we'll illustrate for $m = 2$, $n = 3$

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

clear denominators:

$$b_0 + b_1 s + b_2 s^2 = r_1(s-\lambda_2)(s-\lambda_3) + r_2(s-\lambda_1)(s-\lambda_3) + r_3(s-\lambda_1)(s-\lambda_2)$$

equate coefficients:

- coefficient of s^0:

$$b_0 = (\lambda_2\lambda_3)r_1 + (\lambda_1\lambda_3)r_2 + (\lambda_1\lambda_2)r_3$$

- coefficient of s^1:

$$b_1 = (-\lambda_2 - \lambda_3)r_1 + (-\lambda_1 - \lambda_3)r_2 + (-\lambda_1 - \lambda_2)r_3$$

- coefficient of s^2:

$$b_2 = r_1 + r_2 + r_3$$

now solve for r_1, r_2, r_3 (three equations in three variables)

method 2: to get r_1, multiply both sides by $s - \lambda_1$ to get

$$\frac{(s-\lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

cancel $s - \lambda_1$ term on left and set $s = \lambda_1$:

$$\frac{b_0 + b_1\lambda_1 + b_2\lambda_1^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = r_1$$

an explicit formula for r_1! (can get r_2, r_3 the same way)

in the general case we have the formula

$$r_k = (s-\lambda_k)F(s)\big|_{s=\lambda_k}$$

which means:

- multiply F by $s - \lambda_k$
- then cancel $s - \lambda_k$ from numerator and denominator
- then evaluate at $s = \lambda_k$ to get r_k

example:

$$\frac{s^2-2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

- residue r_1:

$$r_1 = \left(r_1 + \frac{r_2 s}{s+1} + \frac{r_3 s}{s+2} \right)\bigg|_{s=0} = \frac{s^2-2}{(s+1)(s+2)}\bigg|_{s=0} = -1$$

- residue r_2:

$$r_2 = \left(\frac{r_1(s+1)}{s} + r_2 + \frac{r_3(s+1)}{s+2} \right)\bigg|_{s=-1} = \frac{s^2-2}{s(s+2)}\bigg|_{s=-1} = 1$$

- residue r_3:

$$r_3 = \left(\frac{r_1(s+2)}{s} + \frac{r_2(s+2)}{s+1} + r_3 \right)\bigg|_{s=-2} = \frac{s^2-2}{s(s+1)}\bigg|_{s=-2} = 1$$

so we have:

$$\frac{s^2-2}{s(s+1)(s+2)} = \frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

method 3: another explicit and useful expression for r_k is:

$$r_k = \frac{b(\lambda_k)}{a'(\lambda_k)}$$

to see this, note that

$$r_k = \lim_{s \to \lambda_k} \frac{(s - \lambda_k) b(s)}{a(s)} = \lim_{s \to \lambda_k} \frac{b(s) + b'(s)(s - \lambda_k)}{a'(s)} = \frac{b(\lambda_k)}{a'(\lambda_k)}$$

where we used l'Hôpital's rule in second line

example (previous page):

$$\frac{s^2 - 2}{s(s+1)(s+2)} = \frac{s^2 - 2}{s^3 + 2s^2 + 2s}$$

hence,

$$r_1 = \left. \frac{s^2 - 2}{3s^2 + 4s + 2} \right|_{s=0} = -1$$

Example

let's solve

$$v''' - v = 0, \quad v(0) = 1, \quad v'(0) = v''(0) = 0$$

1. take Laplace transform:

$$\underbrace{s^3V(s) - s^2}_{\mathcal{L}(v''')} - V(s) = 0$$

2. solve for V to get

$$V(s) = \frac{s^2}{s^3 - 1}$$

3. the poles of V are the cuberoots of 1, *i.e.*, $e^{j2\pi k/3}$, $k = 0, 1, 2$

$$s^3 - 1 = (s - 1)\left(s + 1/2 + j\sqrt{3}/2\right)\left(s + 1/2 - j\sqrt{3}/2\right)$$

4. now convert V to partial fraction form

$$V(s) = \frac{r_1}{s-1} + \frac{r_2}{s + \frac{1}{2} + j\frac{\sqrt{3}}{2}} + \frac{\overline{r_2}}{s + \frac{1}{2} - j\frac{\sqrt{3}}{2}}$$

to find residues we'll use

$$r_1 = \frac{b(1)}{a'(1)} = \frac{1}{3}, \quad r_2 = \frac{b(-1/2 - j\sqrt{3}/2)}{a'(-1/2 - j\sqrt{3}/2)} = \frac{1}{3}$$

so partial fraction form is

$$V(s) = \frac{\frac{1}{3}}{s-1} + \frac{\frac{1}{3}}{s + \frac{1}{2} + j\frac{\sqrt{3}}{2}} + \frac{\frac{1}{3}}{s + \frac{1}{2} - j\frac{\sqrt{3}}{2}}$$

(check this by just multiplying out . . .)

5. take inverse Laplace transform to get v:

$$\begin{aligned} v(t) &= \frac{1}{3}e^t + \frac{1}{3}e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t} + \frac{1}{3}e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t} \\ &= \frac{1}{3}e^t + \frac{2}{3}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t \end{aligned}$$

6. check that $v''' - v = 0$, $v(0) = 1$, $v'(0) = v''(0) = 0$

Repeated poles

now suppose

$$F(s) = \frac{b(s)}{(s-\lambda_1)^{k_1} \cdots (s-\lambda_l)^{k_l}}$$

- the poles λ_i are distinct $(\lambda_i \neq \lambda_j$ for $i \neq j)$ and have multiplicity k_i
- degree of b less than degree of a

partial fraction expansion has the form

$$\begin{aligned} F(s) &= \frac{r_{1,k_1}}{(s-\lambda_1)^{k_1}} + \frac{r_{1,k_1-1}}{(s-\lambda_1)^{k_1-1}} + \cdots + \frac{r_{1,1}}{s-\lambda_1} \\ &\quad + \frac{r_{2,k_2}}{(s-\lambda_2)^{k_2}} + \frac{r_{2,k_2-1}}{(s-\lambda_2)^{k_2-1}} + \cdots + \frac{r_{2,1}}{s-\lambda_2} \\ &\quad + \cdots + \frac{r_{l,k_l}}{(s-\lambda_l)^{k_l}} + \frac{r_{l,k_l-1}}{(s-\lambda_l)^{k_l-1}} + \cdots + \frac{r_{l,1}}{s-\lambda_l} \end{aligned}$$

n residues, just as before; terms involve higher powers of $1/(s-\lambda)$

example: $F(s) = \dfrac{1}{s^2(s+1)}$ has expansion

$$F(s) = \frac{r_1}{s^2} + \frac{r_2}{s} + \frac{r_3}{s+1}$$

inverse Laplace transform of partial fraction form is easy since

$$\mathcal{L}^{-1}\left(\frac{r}{(s-\lambda)^k}\right) = \frac{r}{(k-1)!}t^{k-1}e^{\lambda t}$$

same types of tricks work to find the $r_{i,j}$'s

- solve linear equations (method 1)
- can find the residues for nonrepeated poles as before

example:

$$\frac{1}{s^2(s+1)} = \frac{r_1}{s^2} + \frac{r_2}{s} + \frac{r_3}{s+1}$$

we get (as before)

$$r_3 = (s+1)F(s)\big|_{s=-1} = 1$$

now clear denominators to get

$$\begin{aligned} r_1(s+1) + r_2 s(s+1) + s^2 &= 1 \\ (1+r_2)s^2 + (r_1+r_2)s + 1 &= 1 \end{aligned}$$

which yields $r_2 = -1$, $r_1 = 1$, so

$$F(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

extension of method 2: to get r_{i,k_i},

- multiply on both sides by $(s-\lambda_i)^{k_i}$
- evaluate at $s=\lambda_i$

gives

$$F(s)(s-\lambda_i)^{k_i}\Big|_{s=\lambda_i} = r_{i,k_i}$$

to get other r's, we have extension

$$\frac{1}{j!}\frac{d^j}{ds^j}\left(F(s)(s-\lambda_i)^{k_i}\right)\Bigg|_{s=\lambda_i} = r_{i,k_i-j}$$

usually the k_i's are small ($e.g.$, 1 or 2), so fortunately this doesn't come up too often

example (ctd.):

$$F(s) = \frac{r_1}{s^2} + \frac{r_2}{s} + \frac{r_3}{s+1}$$

- multiply by s^2:

$$s^2 F(s) = \frac{1}{s+1} = r_1 + r_2 s + \frac{r_3 s^2}{s+1}$$

- evaluate at $s = 0$ to get $r_1 = 1$

- differentiate with respect to s:

$$-\frac{1}{(s+1)^2} = r_2 + \frac{d}{ds}\left(\frac{r_3 s^2}{s+1}\right)$$

- evaluate at $s = 0$ to get $r_2 = -1$

(same as what we got above)

Nonproper rational functions

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n},$$

is called *proper* if $m \leq n$, *strictly proper* if $m < n$, *nonproper* if $m > n$

partial fraction expansion requires *strictly proper* F; to find $\mathcal{L}^{-1}(F)$ for other cases, divide a into b:

$$F(s) = b(s)/a(s) = c(s) + d(s)/a(s)$$

where

$$c(s) = c_0 + \cdots + c_{m-n} s^{m-n}, \quad d = d_0 + \cdots + d_k s^k, \quad k < n$$

then

$$\mathcal{L}^{-1}(F) = c_0 \delta + \cdots + c_{m-n} \delta^{(m-n)} + \mathcal{L}^{-1}(d/a)$$

d/a is strictly proper, hence has partial fraction form

example

$$F(s) = \frac{5s+3}{s+1}$$

is proper, but not strictly proper

$$F(s) = \frac{5(s+1)-5+3}{s+1} = 5 - \frac{2}{s+1},$$

so

$$\mathcal{L}^{-1}(F) = 5\delta(t) - 2e^{-t}$$

in general,

- F strictly proper $\Longleftrightarrow$ f has no impulses at $t=0$
- F proper, not strictly proper $\Longleftrightarrow$ f has an impulse at $t=0$
- F nonproper $\Longleftrightarrow$ f has higher-order impulses at $t=0$
- $m-n$ determines order of impulse at $t=0$

Example

$$F(s) = \frac{s^4 + s^3 - 2s^2 + 1}{s^3 + 2s^2 + s}$$

1. write as a sum of a polynomial and a strictly proper rational function:

$$\begin{aligned}
F(s) &= \frac{s(s^3 + 2s^2 + s) - s(2s^2 + s) + s^3 - 2s^2 + 1}{s^3 + 2s^2 + s} \\
&= s + \frac{-s^3 - 3s^2 + 1}{s^3 + 2s^2 + s} \\
&= s + \frac{-(s^3 + 2s^2 + s) + (2s^2 + s) - 3s^2 + 1}{s^3 + 2s^2 + s} \\
&= s - 1 + \frac{-s^2 + s + 1}{s^3 + 2s^2 + s} \\
&= s - 1 + \frac{-s^2 + s + 1}{s(s+1)^2}
\end{aligned}$$

2. partial fraction expansion

$$\frac{-s^2+s+1}{s(s+1)^2} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{(s+1)^2}$$

- determine r_1:

$$r_1 = \left.\frac{-s^2+s+1}{(s+1)^2}\right|_{s=0} = 1$$

- determine r_3:

$$r_3 = \left.\frac{-s^2+s+1}{s}\right|_{s=-1} = 1$$

- determine r_2:

$$r_2 = \left.\frac{d}{ds}\left(\frac{-s^2+s+1}{s}\right)\right|_{s=-1} = \left.\frac{-s^2-1}{s^2}\right|_{s=-1} = -2$$

(alternatively, just plug in some value of s other than $s=0$ or $s=-1$:

$$\left.\frac{-s^2+s+1}{s(s+1)^2}\right|_{s=1} = \frac{1}{4} = r_1 + \frac{r_2}{2} + \frac{r_3}{4} = 1 + \frac{r_2}{2} + \frac{1}{4} \Longrightarrow r_2 = -2)$$

3. inverse Laplace transform

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\left(s - 1 + \frac{1}{s} - \frac{2}{s+1} + \frac{1}{(s+1)^2}\right) \\ &= \delta'(t) - \delta(t) + 1 - 2e^{-t} + te^{-t}\end{aligned}$$