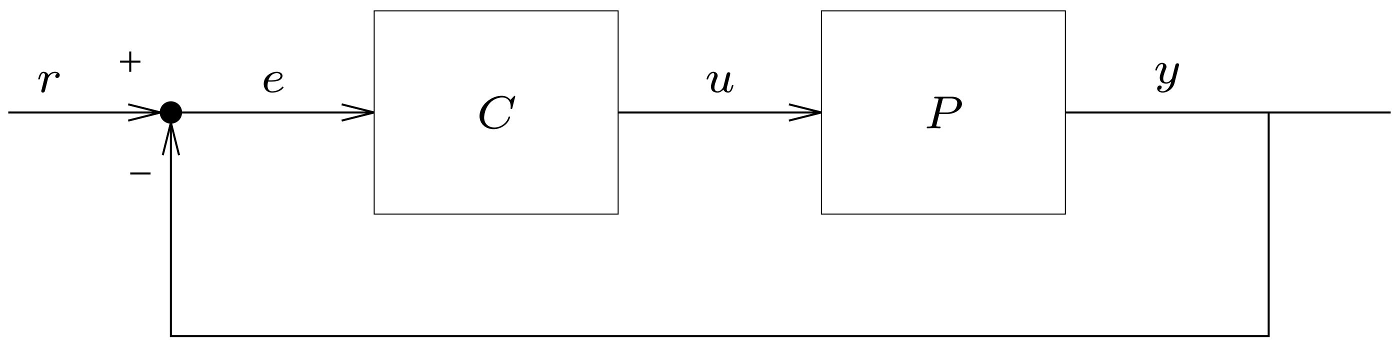
Lecture 14
Integral action

- integral action
- PI control

Proportional control

standard feedback control configuration:



so far we have looked at **proportional control**: $C(s) = k$ is constant

DC sensitivity is $S(0) = 1/(1 + P(0)C(0))$

to make $S(0)$ small, we make $C(0)$ large

Integral action

extreme case: what if $C(0) = \infty$, *i.e.*, C has a **pole** at $s = 0$?

then $S(0)1/(1 + P(0)C(0)) = 0$, which means:

- we have perfect DC tracking: for constant r, $y(t) \to r$ as $t \to \infty$
- for small $\delta P(0)$, $\delta T(0) \approx 0$

but, is it possible? could it ever work?

PI control

C has a pole at $s = 0$ if C has a term like $1/s$ in it, as in

$$C(s) = k_p + \frac{k_i}{s}$$

- called a **proportional plus integral** (PI) control law
- used very widely in practice
- k_p is called the proportional gain; k_i is called the integral gain

PI control law is expressed in time domain as

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)\, d\tau$$

we add another 'corrective' or 'restoring' term, proportional to the *integral* of error a constant error e

- yields a constant corrective reaction (u) for proportional controller
- yields a growing corrective reaction (u) for PI controller

Example

plant from plate heating example:

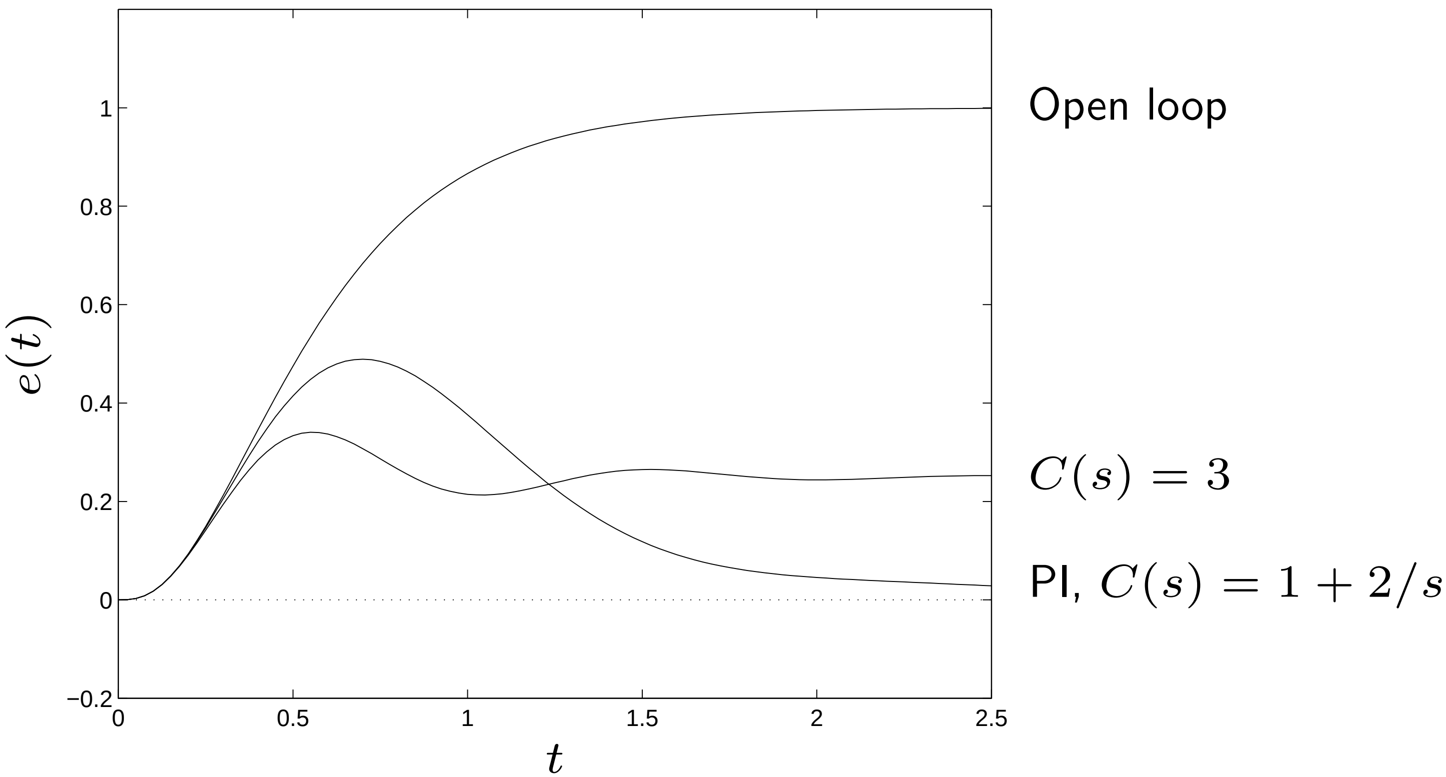
$$P(s) = \frac{1}{(1+0.1s)(1+0.2s)(1+0.3s)},$$

with PI controller $C(s) = 1 + 2/s$

closed-loop transfer function from disturbance power D to temperature error e is then

$$\begin{aligned} \frac{P}{1+PC} &= \frac{\frac{1}{(1+0.1s)(1+0.2s)(1+0.3s)}}{1 + \frac{1}{(1+0.1s)(1+0.2s)(1+0.3s)}(1+2/s)} \\ &= \frac{s}{s(1+0.1s)(1+0.2s)(1+0.3s) + s + 2} \end{aligned}$$

which is stable, with poles -12.2, $-2.32 \pm 3.57j$, -1.51



note that the temperature error e converges to zero

General case

a control system has **integral action** if

- L (*i.e.*, P or C or both) has a pole at $s = 0$
- closed-loop transfer function $T = L/(1 + L)$ is stable

with integral action, $S(0) = \left.\frac{1}{1+L}\right|_{s=0} = 0$ which implies if $r(t)$ is constant,

- $e(t) \to 0$ as $t \to \infty$, *i.e.*, **zero steady-state error**
- $y(t) \to r$ as $t \to \infty$ (called **asymptotic tracking**)

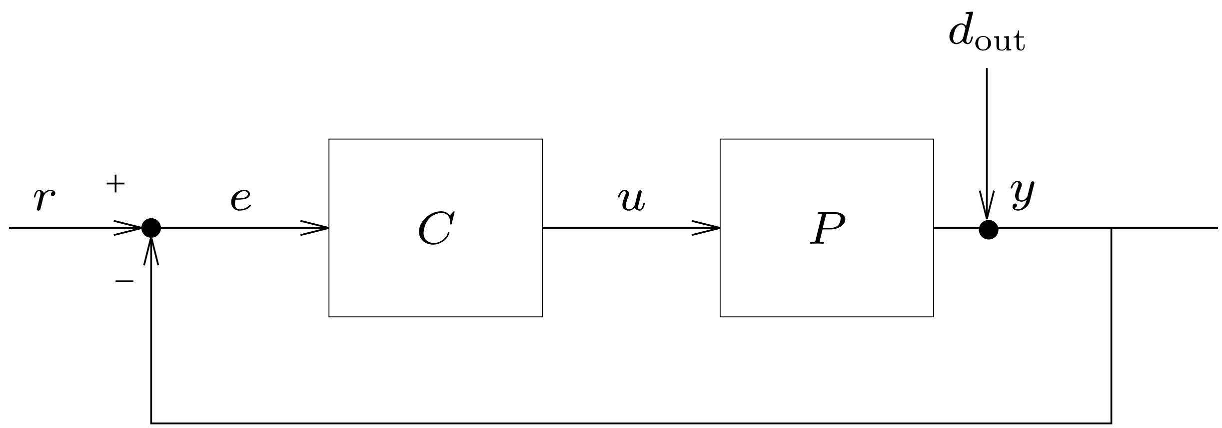
(recall S is the transfer function from r to e)

when r is constant (for long periods of time) it is sometimes called the **set-point** (for y)

integral action $\Rightarrow$ y converges to its set-point

Constant disturbances

consider output disturbance d_{out}:



S is transfer function from d_{out} to e

steady-state error induced by constant d_{out} $S(0)d_{\text{out}} = 0$

i.e., constant output disturbance induces zero steady-state error (called **asymptotic disturbance rejection**)

controller automatically counteracts ('nulls out') any constant disturbance

(constant input disturbances are also rejected if C has a pole at $s = 0$)

Static sensitivity with integral control

suppose

- C has a pole at $s = 0$, but P does not
- $T = L/(1 + L)$ is stable

then we have $T(0) = 1$, regardless of $P(0)$, since

$$T(0) = \left.\frac{P(s)C(s)}{1 + P(s)C(s)}\right|_{s=0}$$

and $C(s) \to \infty$ as $s \to 0$

variations in $P(0)$ have *no effect* on $T(0)$, $i.e.$, $\dfrac{\delta T(0)}{T(0)} = 0$ for *any* (not small) $\delta P(0)$ (as long as T remains stable)

controller pole at $s = 0$ implies: closed-loop DC or steady-state gain is *completely insensitive* to (even large) changes in plant DC gain (as long as T remains stable)

Choice of integral gain

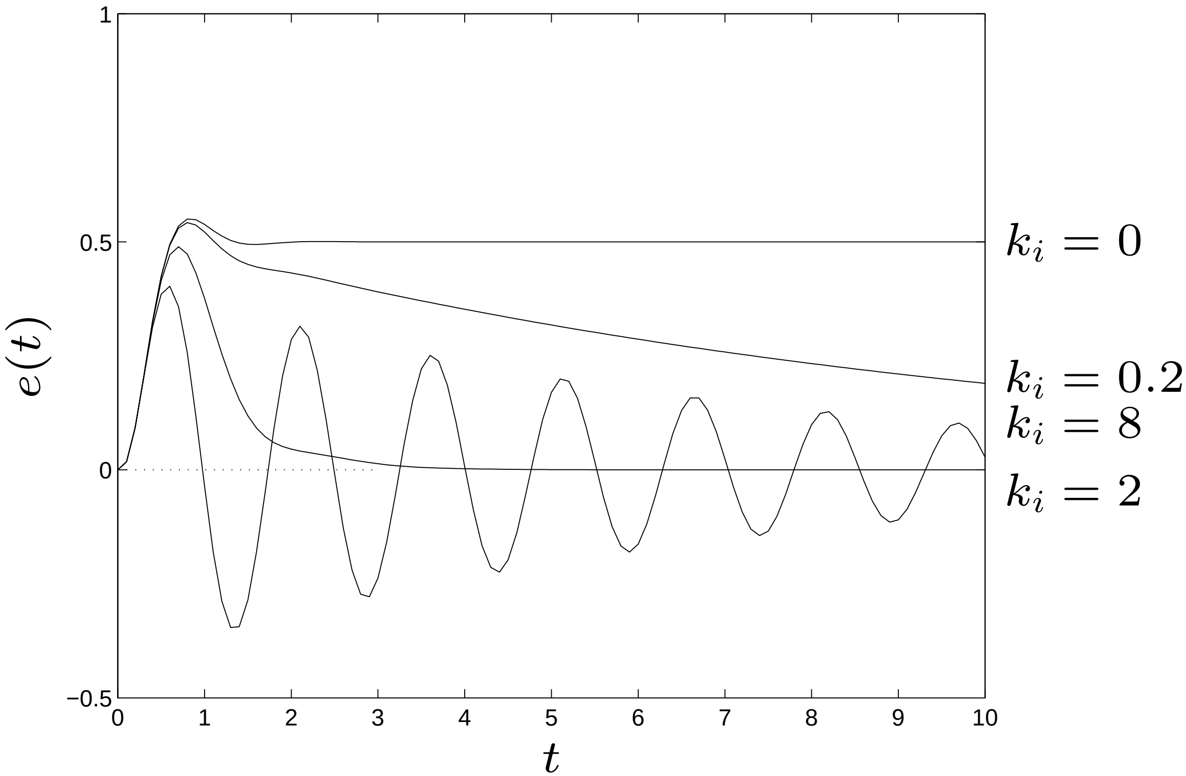
- k_i too small: get asymptotic tracking, disturbance rejection, but only after long time
- k_i too large: oscillatory response, or even instability

(more on choice of k_i later)

closed-loop step responses of heater example, with

$$C(s) = 1 + \frac{k_i}{s},$$

$k_i = 0$ (proportional control; no integral action), $k_i = 0.2,\ 2,\ 8$:



for this example, maybe $k_i \approx 2$ is about right another common form for describing PI controller:

$$C(s) = k\left(1 + 1/(sT_{\text{int}})\right)$$

- T_{int} is called the integral time constant
- $1/T_{\text{int}}$ is called the reset rate

for a constant error e, it takes T_{int} sec for the integral term to equal the proportional term in u

Some plants have pole at $s = 0$, *i.e.*, integration 'built in', *e.g.*,

- $u(t) =$ force on mass; $y(t) =$ position of mass

$$P(s) = \frac{1}{ms^2}$$

- $u(t) =$ voltage applied to DC motor; $y(t) =$ shaft angular velocity

$$P(s) = \frac{k}{Js(1 + sT)}$$

control systems for these plants automatically have integral action

Summary

PI control is widely used in industry

integral action means infinite loop gain at $s = 0$, hence

- zero steady-state tracking error
- zero steady-state effect of constant output disturbance
- zero sensitivity to DC plant gain

(cf. proportional control)

disadvantages of integral action:

- open-loop control system ($i.e.$, L) is unstable — better make sure system doesn't operate open-loop
- excessive integral gain yields closed-loop instability

design tradeoff: how fast we achieve asymptotic tracking vs. stability