EE102 Homework 5 and 6 Solutions

35. The vertical dynamics of a vehicle suspension system, when the vehicle is driving on level ground, are given by

$$(m_v + m_l)d''(t) + bd'(t) + kd(t) = 0.$$

Here

- t is time (in seconds)
- $d(t)$ is the vertical displacement of the vehicle, with respect to its neutral position (in meters)
- $m_v = 10^3$kg is the vehicle mass
- $m_l \geq 0$ (also given in kg) is the mass of the vehicle load (passengers, cargo, etc.)
- $b = 2.2 \cdot 10^4$N/m/s is the suspension damping
- $k = 10^5$N/m is the suspension stiffness

The initial conditions are $d(0) = 0.1$m, $d'(0) = 0$m/s.

What is the *smallest* load mass m_l for which d is oscillatory? (By oscillatory, we mean that $d(t)$ passes through zero infinitely many times.)

Solution. Let $m = m_v + m_l$. Then, taking the Laplace transform of the given differential equation, we obtain

$$ms^2D(s) - msd'(0) - md(0) + bsD(s) - bd(0) + kD(s) = 0$$

The above equation can be rewritten as

$$D(s) = \frac{msd'(0) + (m + b)d(0)}{ms^2 + bs + k}$$

The system will exhibit oscillatory behavior when the roots of the characteristic polynomial $ms^2 + bs + k$ are complex. The roots are given by the solution of the quadratic equation, *i.e.*,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

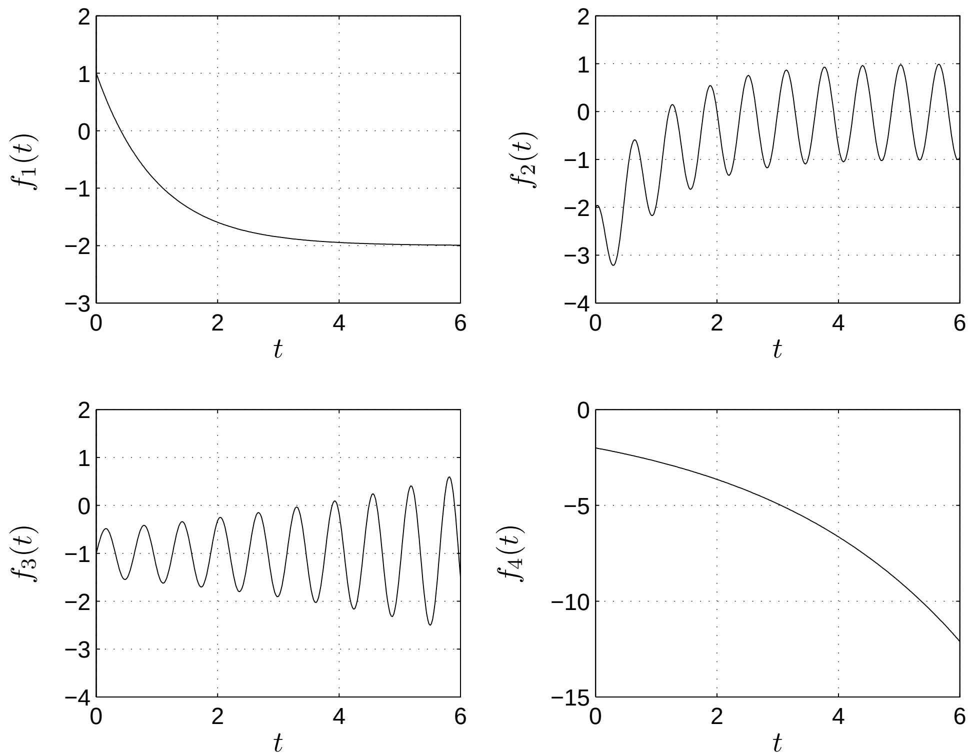
Therefore, the roots will be complex when $b^2 - 4mk < 0$. We can rewrite this condition as

$$m > \frac{b^2}{4k} = 1210$$

Finally, since $m = m_v + m_l$ and $m_v = 1000$, we obtain that $m_l > 210$ for d to become oscillatory.

41. Four functions f_1, f_2, f_3, and f_4, are shown below. Their Laplace transforms are F_1, F_2, F_3, and F_4, respectively. You can assume $F_1, \ldots, F_4$ are rational functions, with no more than three poles (counting multiplicities).

Please note carefully the vertical scales — they are *not* the same!



Estimate the poles of $F_1, \ldots, F_4$, using the smallest number needed to give a reasonable match. If you can get a reasonable match with one pole, then give just one pole; if two poles suffice then give just two. Give three poles only if three poles are required to match the given f_i.

- We want *specific numbers*, not just qualitative answers such as 'one positive real, one complex pair' or $\sigma + j\omega$ (without specifying σ or ω). Make clear indications on the plots how you got the numbers.
- Give complex poles separately, as in '$1+j$, $1-j$'; we will *not* automatically supply conjugates of complex numbers.
- Give multiple poles by repeating them in your answers, *e.g.*, '-3, -1, -1' (meaning, one pole at $s=-3$, and another pole of multiplicity two at $s=-1$).

Solution:

(a) From the graph of f_1, we see that there is no sinusoidal ringing, and something like only one exponential decay towards a nonzero value, -2. Therefore, we have only 2 real poles: one corresponding to the decaying term and one corresponding to the constant term. The cosntant term corresponds to a pole at $s=0$. The exponentially decaying term starts with amplitude $1-(-2)=3$, and it reaches 37% of the amplitude in approximately 1sec. Therefore, that pole is at -1.

So the poles are: -1, 0.

(b) From the plot of f_2, we note that the waveform exhibits some sinusoidal ringing, and the amplitude of the ringing is approximately constant, neither growing nor decaying. This suggests a complex pole pair along the imaginary axis. Furthermore, when we draw a line down the middle of the oscillations we find that there is also a real, exponentially decaying term with a negative coefficient. So, the third pole is real and negative.

Let's find approximate numerical values for the poles. We see around 9.5 periods of the signal in 6 time units, so the oscillation frequency is roughly $\omega \approx \frac{2\pi}{6/9.5} \approx 10$. So we have the poles $\pm \jmath 10$. As for the third pole, we see that it decays to 37% of the amplitude in approximately 1sec. Therefore, the third pole is at -1.

The poles are: $10\jmath$, $-10\jmath$, -1.

(c) From the plot of f_3, we see a sinusoidal ringing with an exponentially growing amplitude, which suggests a complex pole pair in the right half plane. The oscillation frequency of the ringing is the same as in f_2, *i.e.*, $\omega = 10$. At time zero, the amplitude of the ringing is about 0.5. At 4 seconds, the amplitude is, say, 1.1 or 1.2. Thus we have $e^{\sigma 4} = 1.2$, so $\sigma \approx \log(1.2/0.5)/4 \approx 0.2$ or so. Therefore, the poles are $0.2 \pm \jmath 10$.

Drawing a line down the middle of the oscillation indicates that there is also a constant offset of -1, which corresponds to a pole at $s = 0$.

The poles are: $0.2 + 10\jmath$, $0.2 - 10\jmath$, 0.

(d) In the last plot, we see no sinusoidal ringing; We see only one exponential growth (multiplied by negative coefficient). At time zero, $f_4(t)$ is about -2; at time equals 6, $f_4(t)$ is about -12. So, the growth rate is approximately $\log(-12/-2)/6 \approx 0.3$. So, the pole is at 0.3.

55. *Reducing the rise-time of a signal.* In a certain digital system a voltage signal should ideally switch from 0V to 5V infinitely fast, *i.e.*, with zero rise-time. But due to the finite bandwidth of the electronics that generates the signal, it has the form

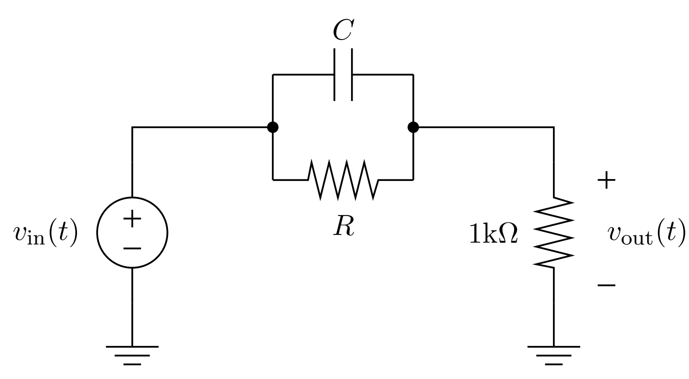
$$v_{\text{in}}(t) = 5\left(1 - e^{-t/T}\right) \quad \text{for } t \geq 0$$

where $T = 1\mu$sec. Thus, the signal has a rise-time around a few μsec.

An engineer claims that the circuit shown below can be used to reduce the rise-time of the signal, provided the component values R and C are chosen correctly. Specifically, the engineer claims that by choosing R and C correctly, we can have

$$v_{\text{out}}(t) = a\left(1 - e^{-10t/T}\right) \quad \text{for } t \geq 0$$

where a is some nonzero constant. Thus, the rise-time of v_{out} is a factor of 10 smaller than the rise-time of v_{in}, *i.e.*, a few hundred nsec.



Here is the problem: determine whether the engineer's claim is true or false. If the claim is true, find specific, numerical values of R and C that validate the claim. If the claim is false, briefly explain why the engineer's idea will not work.

(You can assume the circuit starts in the relaxed state, *i.e.*, no charge on the capacitor. And no, you cannot use negative R or C.)

Solution:
First we'll find the relationship between $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$.

$$V_{\text{out}} = V_{\text{in}} \frac{1000}{1000 + R \parallel \frac{1}{sC}} = V_{\text{in}} \frac{1000(1 + sRC)}{(1000 + R) + s1000RC}$$

Now let's assume that the engineer's claim is true, *i.e.*, for some value of R and C we have $v_{\text{out}}(t) = a(1 - e^{-10t/T})$. Then:

$$V_{\text{out}}(s) = \frac{a}{s} - \frac{a}{s + \frac{10}{10^{-6}}} = \left(\frac{5}{s} - \frac{5}{s + \frac{1}{10^{-6}}} \right) \frac{s + 1/RC}{s + \frac{1000+R}{1000RC}}$$

Simplifying, we get

$$\frac{10^7 a}{s + 10^7} = \left(\frac{5 \cdot 10^6}{s + 10^6} \right) \left(\frac{s + 1/RC}{s + \frac{1000+R}{1000RC}} \right).$$

We need to choose R and C to make this equality hold.

But how can this be? The left hand side has only one pole, while the right hand side has two ... The *only* way this can happen is if the zero on the right hand side, at $s = -1/RC$, cancels the pole at -10^6, *i.e.*, we take $1/RC = 10^6$. This yields

$$\frac{10^7 a}{s + 10^7} = \frac{5 \cdot 10^6}{s + \frac{1000+R}{10^{-3}}}.$$

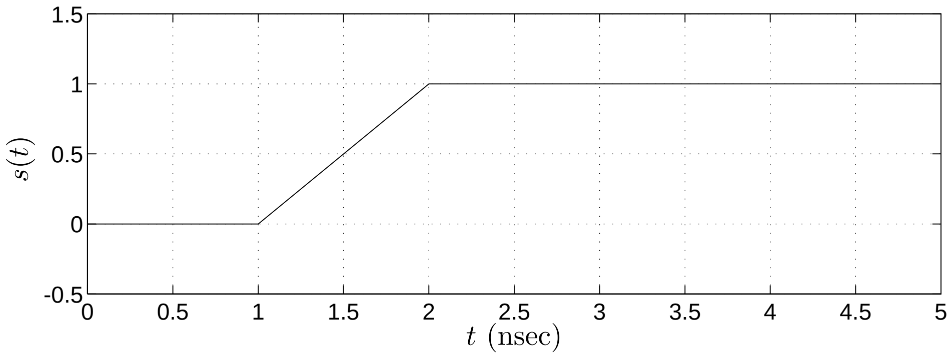
Now we want the remaining pole to be at $s = -10^7$, so we have $(1000 + R)/(10^{-3}) = 10^7$. This yields $R = 9\text{k}\Omega$, so $C = 111\text{pF}$.

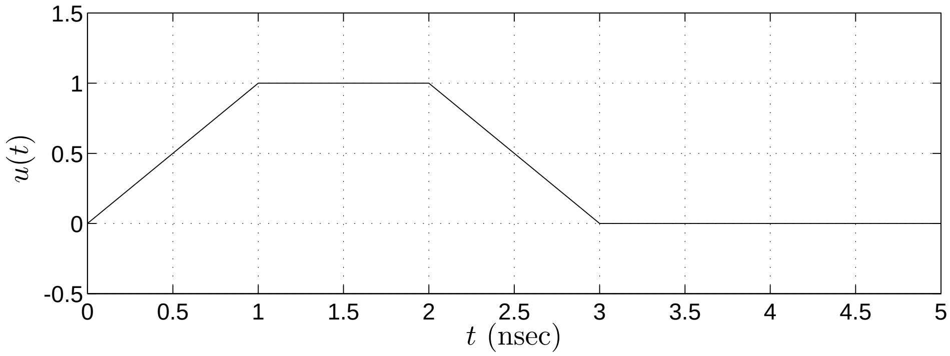
So we see the engineer's claim is true.

Note that $a = 0.5\text{V}$. Thus we see one disadvantage of this circuit — the rise-time is ten times faster but the signal level has been cut by a factor of 10!

Let's also mention another method of solving this problem. Once we have V_{out} and V_{in}, we divide to find out what the transfer function must be to make the claim true. Then we simply see whether we can choose the parameters R and C to make the transfer function of our circuit equal to the one we require, *i.e.*, $V_{\text{out}}/V_{\text{in}}$. (Of course you get the same answer.)

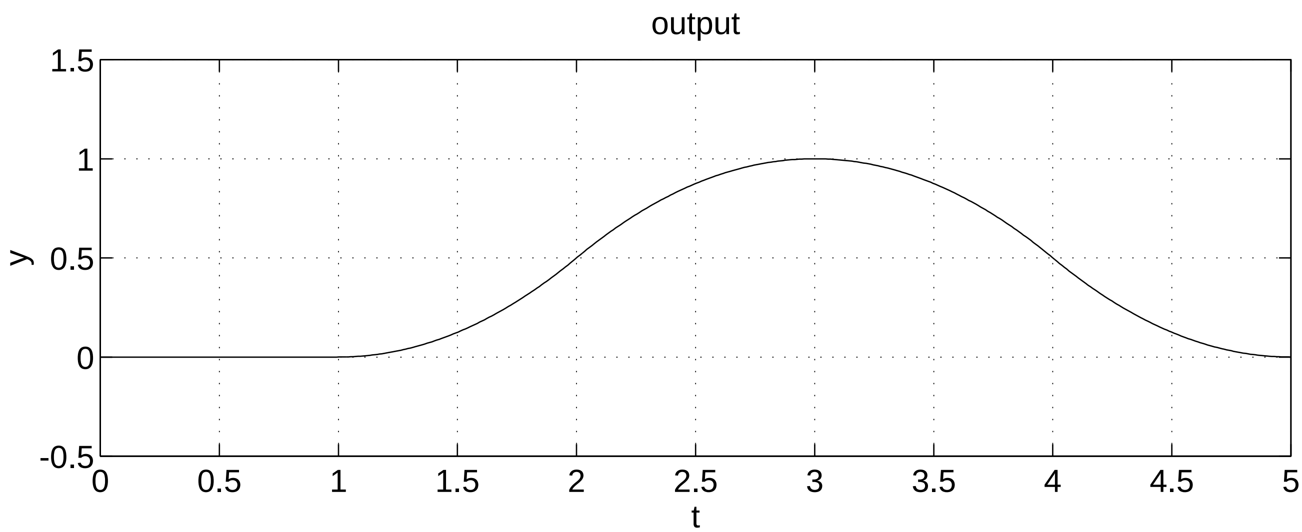
56. The top plot below shows the step response of a system described by a transfer function. Below that is a plot of an input $u(t)$ that we apply to this system. Sketch the response (output) $y(t)$.





Solution:
Perhaps the easiest way to solve this problem is to differentiate the step response to get the impulse response, then convolve the impulse response with the input to find the output. The impulse response is zero except between $t = 1$nsec and $t = 2$nsec, where it has the value 10^9 (note the time axis; if you just ingore the units everything works out OK anyway!).

Thus the output at time t is just the integral of the input between $t-2$ and $t-3$ nanoseconds ago, multiplied by 10^9. It is zero until $t = 1$; then it ramps up quadratically (since we are integrating a ramp) until $t = 2$. Between $t = 2$ and $t = 3$ we integrate part of a ramp and part of the constant 1; this yields another quadratic. The maximum value is achieved right at $t = 3$, when we integrate the constant 1, so the output is 1. In the next two nanoseconds the same thing happens, backwards in time. The final solution is shown below:



If you don't get that fluffy derivation, or simply must see formulas to believe, here is another derivation. The step response evidently has the form

$$s(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1 & t \ge 2 \end{cases}$$

where t is in nanonseconds. Differentiate to get the step response h:

$$h(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

The input u has the form:

$$u(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ t-3 & 2 \le t \le 3 \\ 0 & t \ge 3 \end{cases}$$

Hence the output y is just:

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau = \int_{t-2}^{t-1} u(\tau)\,d\tau.$$

The output will have 5 separate segments, *i.e.*, different formulas, depending on t. We'll just work out an interesting case, and let you check the others. We'll find $y(t)$ for $2 \le t \le 3$. From our formula above,

$$y(t) = \int_{t-2}^{t-1} u(\tau)\,d\tau = \int_{t-2}^{1} \tau\,d\tau + \int_{1}^{t-1} 1\,d\tau = \frac{1}{2}(1-(t-2)^2) + t - 2$$

which corresponds to the quadratic that starts at value $1/2$ for $t = 2$ and ends up at value 1 (and slope zero) at $t = 1$.

You could also work this problem using Laplace transforms, but it's not a pretty sight. The transfer function is

$$H(s) = \frac{e^{-s} - e^{-2s}}{s}$$

and the Laplace transform of u is

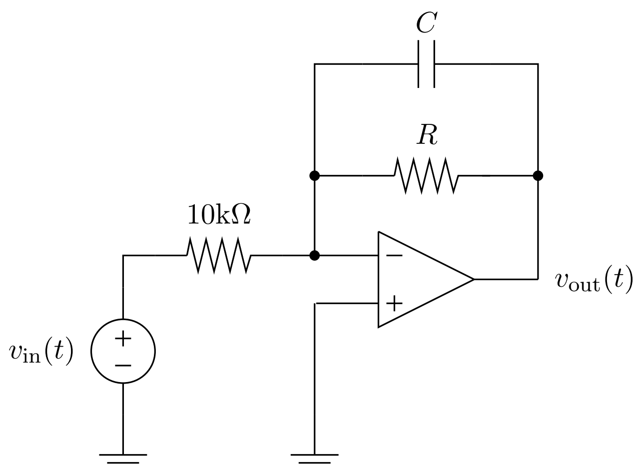
$$U(s) = \frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}.$$

Thus we have

$$\begin{aligned} Y(s) &= H(s)U(s) \\ &= \frac{e^{-s} - e^{-2s}}{s} \frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2} \\ &= \frac{e^{-s} - 2e^{-2s} + 2e^{-4s} - e^{-5s}}{s^3} \end{aligned}$$

which you will instantly recognize as the Laplace transform of our solution shown in the plot.

65. The circuit below is a simple one-pole lowpass filter.



Find (positive) R and C such that:

- The (magnitude of the) DC gain is +12dB.
- The magnitude of the transfer function at the frequency 1kHz is 3dB less than the magnitude of the DC gain.

You can assume the op-amp is ideal. Give numerical values for R and C. An acurracy of 10% will suffice.

Solution.

First, let's find the transfer function from v_{in} to v_{out}. You could use standard circuit analysis, but by now you should just recognize this circuit as an inverting amplifier and know that the transfer function is given by

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R \| \frac{1}{sC}}{10\text{k}\Omega} = -\frac{R/10\text{k}\Omega}{1 + sRC}$$

This circuit has only one pole, at $s = -1/RC$. The DC gain is found by setting $s = 0$: $H(0) = -R/10\text{k}\Omega$. (You can also see this by looking at the circuit with the capacitor removed.)

We are given that the (magnitude of the) DC gain is +12dB, which is roughly 4 (6dB + 6dB; each is a factor of around 2). Thus, $|H(0)| = R/10\text{k}\Omega = 4$, so we must have $R = 40\text{k}\Omega$. (Note that the DC gain is actually negative, *i.e.*, $H(0) = -4$.)

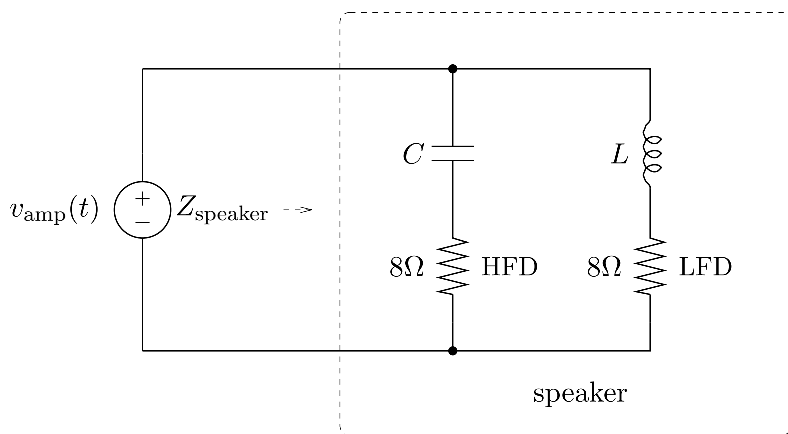
The gain is 3dB less at 1kHz, which means that we have a pole at $s = -1\text{kHz} = -2000\pi$. Since the pole is at $s = -1/RC$, we have:

$$C = \frac{1}{40\text{k}\Omega \cdot 2000\pi} = 4\text{nF}.$$

68. *A simple two-way crossover circuit.* A typical high-fidelity speaker has separate drivers for low and high frequencies. (The driver is the physical device that vibrates to create the sound you hear. The old terms for the low and high frequency drivers are *woofer* and *tweeter*, respectively.)

The circuit shown below, called a *speaker crossover network*, is used to divide the audio signal coming from the amplifier into a low frequency part for the low frequency driver (LFD) and a high frequency part for the high frequency driver (HFD). Since the audio spectrum is divided into two parts, this is called a two-way system (three-way are also common).

The amplifier is modeled as a voltage source (which is a very good model), and the low and high frequency drivers are modeled as 8Ω resistances (which is not a good model of real drivers, but we will use it for this problem).



The crossover network is designed so that the transfer function from the amplifier to each driver has magnitude −3dB at a frequency ω_c called the *crossover frequency* of the speaker.

(a) Choose C and L so that the crossover frequency is 2kHz. Do this carefully as you will need your answers in part b.

(b) Using the values for L and C found in 8a and 8b, find $Z_{\text{speaker}}(s)$, the impedance of the two-way speaker seen by the amplifier (as indicated in the schematic).

Solution:
We can separate the analysis for the high frequency driver (HFD) and low frequency driver (LFD) since the two circuits are driven in parallel by a voltage source. Notice the HFD is just a simple RC high-pass filter and LFD is just a simple RL low-pass filter!

For the HFD, we have the transfer function:

$$\frac{V_{\mathrm{HFD}}}{V_{\mathrm{amp}}} = \frac{8}{8+\frac{1}{sC}} = \frac{8sC}{1+8sC}$$

Note that crossover frequency is the frequency at which the transfer function is -3dB, *i.e.*, it is the (absolute value of the) pole location. With the crossover frequency at 2kHz, we have

$$\frac{1}{8C} = (2\pi)(2000)$$

So $C = 9.95\mu$F.

Similarly, for the LFD we have the transfer function:

$$\frac{V_{\mathrm{LFD}}}{V_{\mathrm{amp}}} = \frac{8}{8+sL} = \frac{1}{1+s\frac{L}{8}}$$

With the crossover frequency at 2kHz, we have

$$\frac{8}{L} = (2\pi)(2000)$$

So $L = 0.64$mH.

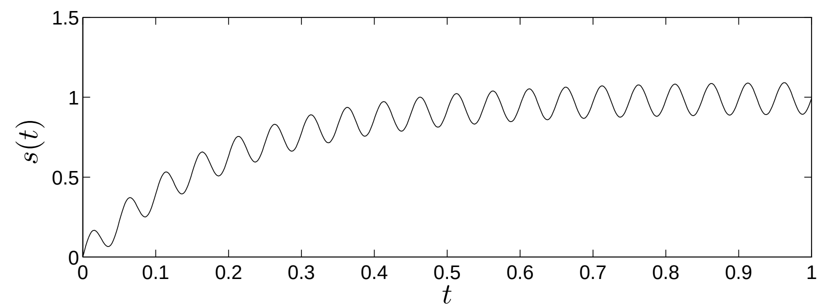
Let's denote the resistance of the HFD and LFD as R. Now since the poles of the HFD and the LFD have already been designed to be 2kHz, we have $RC = L/R$. Therefore,

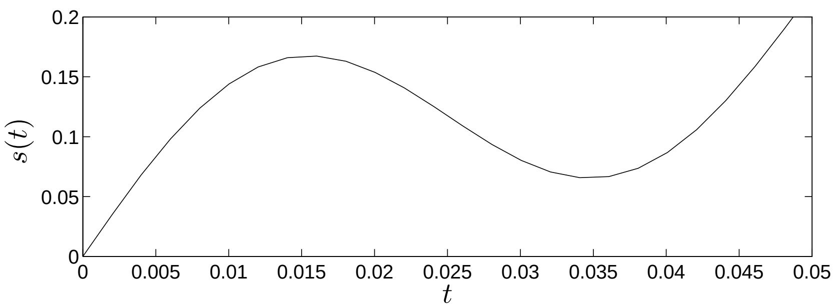
$$Z_{\mathrm{speaker}}(s) = \frac{(R+\frac{1}{sC})(R+sL)}{R+\frac{1}{sC}+R+sL} = \frac{(R+\frac{R^2}{sL})(R+sL)}{2R+\frac{R^2}{sL}+sL} = R.$$

So we see that $Z_{\mathrm{speaker}}(s)$ is constant — it is simply 8Ω, independent of frequency. To the amplifier, the speaker looks like a simple resistance of 8Ω.

In practice, real drivers do not have constant 8Ω impedance, and also, 2 and 3 pole crossover networks are more common than the simple 1 pole network described in this problem. But the idea is the same.

72. The unit step response $s(t)$ of a system described by a transfer function H, which has three poles, is shown in the two plots below. The two plots have different ranges; the second plot allows you to see details for small t.





(a) Estimate the poles. An accuracy of $\pm 20\%$ is acceptable.

(b) At high frequencies $|H(j\omega)|$ becomes small. From the data given, can you determine the rate at which it decreases for large frequency (*e.g.*, 12 db/octave)? Either give the rate (in dB/octave) or state "cannot determine" if the data given is not sufficient to determine the high-frequency rolloff rate.

Solution:
We start by estimating the step response. We should see evidence of the three poles of the transfer function H, along with a pole at $s = 0$ (*i.e.*, a constant) associated with the step input. There is a sinusoidal term that does not decay or grow, with an amplitude of about 0.1. The frequency of this term is easy to tell: two cycles fit in 0.1sec so the frequency is $40\pi\text{rad}/sec$. Thus there are two poles at $s = \pm j40\pi \approx 126\text{rad/sec}$. Drawing a line through the average value of this oscillation we see that something like $1 - e^{-4t}$ remains. This corresponds to a pole at $s = 0$ (which comes from the step input) and $s = -4$. Thus the poles of H must be about $-4, \ \pm j126$.

There are several ways to find the rate at which $|H(j\omega)|$ decreases for large ω. Perhaps the easiest way is to remember the connection between the behavior of the step response for small t and the rolloff rate of the frequency response. The step response has an initial nonzero slope, so H must have one more pole than zero, *i.e.*, two zeros. This means that the rolloff rate is 6dB/octave.

We can also find the rate by directly estimating the transfer function H from the step response plot. The step response has the (approximate) form

$$s(t) \approx 1 - e^{-4t} + \cos(126t + \phi).$$

We can figure out the phase of the sinusoid by examining the lower plot, and noting that $s(0) \approx \cos(\phi)$. This suggests that the step response is

$$s(t) \approx 1 - e^{-4t} + \sin(126t).$$

(Actually this method is extremely prone to approximation error.) It's Laplace transform is therefore

$$S(s) \approx \frac{1}{s} - \frac{1}{s+4} + \frac{1}{s^2+126^2} = \frac{5s^2+4s+4\cdot 126^2}{s(s+4)(s^2+126^2)}.$$

Therefore

$$H(s) = sS(s) \approx \frac{5s^2+4s+4\cdot 126^2}{(s+4)(s^2+126^2)}$$

which has three poles and two zeros. Hence for large ω, the frequency response rolls off at 6dB/octave.

74. *Transfer function from rainfall to river height.* The height of a certain river depends on the past rainfall in the region. Specifically, let $u(t)$ denote the rainfall rate, in inches-per-hour, in a region at time t, and let $y(t)$ denote the river height, in feet, above a reference (dry period) level, at time t. The time t is measured in hours; we'll only consider $t \geq 0$.

Analysis of past data shows that the relation between rainfall and river height can be accurately described by a transfer function:

$$Y(s) = H(s)U(s), \quad H(s) = \frac{10}{(3s+1)(30s+1)}$$

(You don't need to know any hydrology to do this problem, but you might be interested in the physical basis of this two-pole transfer function. The fast pole is due to runoff from surface water and small tributaries, which contribute a relatively small amount of water relatively quickly. The slow pole is due to flow from larger tributaries and deeper ground water, which contribute more water into the river, over a much longer time scale.)

A brief but intense downpour. (Parts a and b.) Suppose that after a long dry spell (*i.e.*, no rain) it rains intensely at 12 inches-per-hour, for 5 minutes. This causes the river height to rise for a while, and then later recede.

(a) How long does it take, after the beginning of the brief downpour, for the river to reach its maximum height? We'll denote this delay as $t_{\max}$ (in hours).

(b) What is the maximum height of the river? We'll denote this maximum height as $y_{\max}$ (in feet).

Note: you can make a reasonable approximation provided you say what you are doing.

A continual rain. (Parts c and d.) Suppose that after a long dry spell it starts raining continuously at a rate of 1 inch-per-hour (and doesn't stop). This causes the river height to rise.

(c) What is the ultimate height of the river, *i.e.*, $y_{\text{ult}} = \lim_{t\to\infty} y(t)$?

(d) A flood occurs when the river height $y(t)$ reaches 8 feet. How long will it take, after the onset of the steady rain, to reach flood condition? We'll denote this time as t_{flood}. If the river never reaches 8 feet, give your answer as 'never'.

Note: you can make a reasonable approximation provided you say what you are doing.

Solution:

(a) First, we notice that the time duration of the downpour (5 min) is small in comparison to the time constants of the poles (3 hours and 30 hours). Hence we can reasonably approximate the intense downpour as an impulse function. The area under the rainfall $u(t)$ is (12 inches/hour)(1/12 hour), *i.e.*, 1 (inch). So $u(t) \approx \delta(t)$, and consequently the river height y is (approximately) given by the impulse response of the transfer function. (If you model the downpour more accurately as a step function which turns on at $t = 0$ and off and $t = 1/12$, the mathematic gets much more involved, and the final answer is nearly the same.)

Thus $Y(s) = H(s)$, so we use a partial fraction expansion:

$$Y(s) = \frac{10}{(3s+1)(30s+1)} = \frac{-0.37}{s+1/3} + \frac{0.37}{s+1/30}$$

The inverse transform is then $y(t) = 0.37(e^{-t/30} - e^{-t/3})$.

This is a good point at which to do a 'reasonableness' check. This $y(t)$ starts at 0, rises for a while, then decays — which does seem very reasonable!

We find the maximum of $y(t)$ by setting the derivative to zero:

$$0.37(\frac{-1}{30}e^{-t/30} + \frac{1}{3}e^{-t/3}) = 0$$

which can be solved to yield $t = 7.675$. Thus the river height peaks $t_{\text{max}} = 7.7$ hours later. This answer seems reasonable; the river peaks about at double the fast pole's time constant.

(b) The maximum height of the river is found by substituting $t = 7.675$ hours into our expression for $y(t)$ found above. We find $y_{\text{max}} = 0.257$ feet $= 3.09$ inches. Again, this seems to correlate well with our intuitive feeling for a river's behavior. A brief downpour occurs, and the river rises 3 inches about 7 hours later.

(c) The continual rain is just a unit step function, so the river height is just the step response of the transfer function:

$$Y(s) = \frac{10}{(3s+1)(30s+1)} \cdot \frac{1}{s} = \frac{10}{s} + \frac{10/9}{s+1/3} - \frac{100/9}{s+1/30}$$

Hence $y(t) = 10 + \frac{10}{9}e^{-t/3} - \frac{100}{9}e^{-t/30}$. From inspection, we see that as $t \to \infty$, $y \to 10$ feet!

Another way to solve this problem, which is probably easier, is to notice that we need to find the limit of the step response, which is just the DC gain of the transfer function. Plugging in $s = 0$ yields, again, the answer $y_{\text{ult}} = 10$.

(d) The flood level is 8 feet, less than the final height of 10 feet we solved for in part (1c). So we do reach flood condition; it's just a question of when. Using our expression for y found above, the flood occurs for t with

$$8 = 10 + \frac{10}{9}e^{-t/3} - \frac{100}{9}e^{-t/30}.$$

In fact, it is not so easy to solve this equation! Now is a very good time for a simple approximation, as our note suggests. Note that the term $\frac{10}{9}e^{-t/3}$ decays quickly, within say 10 hours, after which the river height will be approximately given by $10 - \frac{100}{9} e^{-t/30}$. Thus we can find an approximate solution by solving

$$8 = 10 - \frac{100}{9} e^{-t/30}$$

which yields $t = 51.4$ hours. (Note that our assumption that the first term is small is certainly correct when $t = 51.4$ hours; $e^{-51.4/3}$ is quite small!)

Hence we predict disaster crews have about 2 days to prepare for flooding after a big storm hits! (Provided the rain continues at 1 inch/hour.)