EE102 Homework 2, 3, and 4 Solutions

7. *Some convolution systems.* Consider a convolution system,

$$y(t) = \int_{-\infty}^{+\infty} u(t - \tau)h(\tau) \, d\tau,$$

where h is a function called the *kernel* or *impulse response* of the system.

(a) Suppose the input is a unit impulse function, *i.e.*, $u = \delta$. What is the output y? (This explains the terminology above.)

(b) Suppose $h = \delta$. What does the system do?

(c) Suppose h is a unit step function. What does the system do?

(d) Suppose $h = \delta'$. What does the system do?

(e) Suppose $h(t) = \delta(t - 1)$. What does the system do?

(f) Suppose h is a rectangular pulse signal that is one between 0 and 1. What does the system do?

Solution:
To solve these problems we just plug the given u or h into the convolution formula.

(a) If $u = \delta$ then we have

$$y(t) = \int_{-\infty}^{+\infty} \delta(t - \tau)h(\tau) \, d\tau = h(t).$$

In other words, the output y is the same as h. That's why we call h the impulse response; it's what comes out if an impulse goes in.

(b) If we have $h = \delta$, we find that $y(t) = u(t)$. In words, this system does nothing: its output is exactly the same as its input.

(c) If h is a unit step function, which we'll denote $h(t) = \mathbf{1}(t)$ then we have

$$y(t) = \int_{-\infty}^{+\infty} u(t - \tau)\mathbf{1}(\tau) \, d\tau = \int_{0}^{\infty} u(t - \tau) \, d\tau = \int_{-\infty}^{t} u(\tau) \, d\tau.$$

(We changed variables in the last expression.) In words, the output is the integral of the input; this system is just an integrator.

(d) If $h = \delta'$, then we have

$$y(t) = \int_{-\infty}^{+\infty} u(t - \tau)\delta'(\tau) \, d\tau$$

Integrating by parts, we find

$$y(t) = u(t - \tau)\delta(\tau)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d}{d\tau}(u(t - \tau))\delta(\tau) \, dt = \int_{-\infty}^{+\infty} u'(t - \tau)\delta(\tau) \, dt = u'(t)$$

Since $y(t) = u'(t)$, this system is a differentiator.

(e) If $h(t) = \delta(t-1)$ then the output is delayed, i.e., $u(t-1)$.

$$y(t) = \int u(t-\tau)\delta(\tau-1)\, d\tau = u(t-1).$$

This system is a 1-second delay.

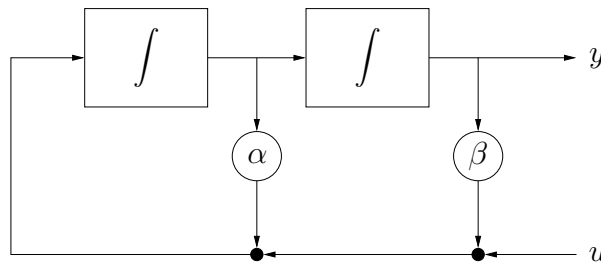(f) If h is a rectangular pulse signal that is one between 0 and 1, then we have

$$y(t) = \int_0^1 u(t-\tau)\, d\tau$$

Changing variables we can write this as

$$y(t) = \int_{t-1}^t u(\tau)\, d\tau.$$

This shows that output is the integral (or in this case average) of the input over the last second.

8. Describe the system shown below as an LCCODE. *Hint:* first label all signals, then write down how they are related.

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Solution:
The signal going into the righthand integrator is y', and similarly, the signal going into the lefthand integrator is y''. The signal at the far left of the system, which is y'', consists of the sum of three terms: $\alpha y'$, βy, and u. We can write this out as an equation: $y'' = \alpha y' + \beta y + u$. This can be expressed as the LCCODE

$$y'' - \alpha y' - \beta y = u.$$

10. *Block diagram from equations.* An interconnected set of systems is described by the following equations:

$$v = A(u - v), \quad w = B(v - z), \quad z = C(w).$$

Here u, v, w, z are signals and A, B, C are systems. You can consider u as the external input to the interconnected systems, and z as the external output of the interconnected system.

(a) Draw a pretty block diagram representing these equations. *Hint:* it usually takes two or three passes to get a pretty block diagram.

(b) Now suppose that the systems A, B, C are scaling systems with gains a, b, c, respectively. Express z in terms of u. (In other words, eliminate the signals v and w from the equations.)

Solution:

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(a) There are a lot of ways to draw this block diagram, here's one that is pretty attractive:
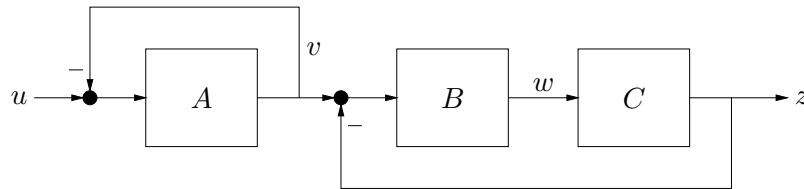
(b) First, using the first equation given and solving for v in terms of u:

$$v = \frac{A}{1+A}u$$

Next, using the expression for w given in the second equation, and substituting it into the third given equation, we can write an equation for z in terms of v:
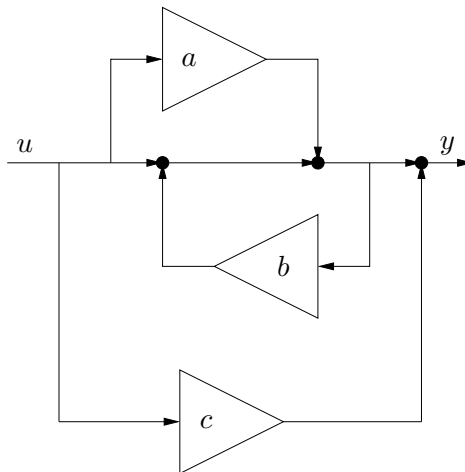
$$z = BC(v - z)$$

$$z = \frac{BC}{1+BC}v$$

Now, substituting in the equation for v in terms of u from before, we find:

$$z = (\frac{BC}{1+BC})(\frac{A}{1+A})u$$

11. Consider the block diagram shown below:

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The triangle shaped blocks represent scaling systems with gains a, b, and c.

Find a *simple* mathematical expression for the output y in terms of the input u and the constants a, b, and c. Your answer should be *only* in terms of the input u and the scale factors a, b, and c.

Solution. Starting from the output, we have:

$$
\begin{aligned}
y &= cu + y - cu \\
&= cu + au + u + b(y - cu) \\
&= (c + a + 1 - bc)u + by
\end{aligned}
$$

Solving the above equation for y, we obtain:

$$
y = \frac{c + a + 1 - bc}{1 - b} u.
$$

13. Find the Laplace transform of the following functions.

    (a) $f(t) = (1 + t - t^2)e^{-3t}$.

    (b) $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ -1 & 2 \le t \end{cases}$

    (c) $f(t) = 1 - e^{-t/T}$ where $T > 0$.

Solution:
Finding $F(s)$ is simply a matter of integration:

    (a)

$$
\begin{aligned}
F(s) &= \int_0^\infty (1 + t - t^2)e^{-3t}e^{-st} dt \\
&= \int_0^\infty e^{-(s+3)t} + te^{-(s+3)t} - t^2 e^{-(s+3)t} dt \\
&= \frac{1}{s+3} + \frac{1}{(s+3)^2} - \frac{2}{(s+3)^3}
\end{aligned}
$$

    (b)

$$
\begin{aligned}
F(s) &= \int_1^2 e^{-st} + \int_2^\infty -e^{-st} \\
&= \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s} \\
&= \frac{1}{s}\left[e^{-s} - 2e^{-2s}\right]
\end{aligned}
$$

    (c)

$$
\begin{aligned}
F(s) &= \int_0^\infty \left(1 - e^{-t/T}\right)e^{-st} dt \\
&= \int_0^\infty \left(e^{-st} - e^{-(s+1/T)t} dt\right) dt \\
&= \frac{1}{s} - \frac{T}{sT+1}
\end{aligned}
$$

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These can also be solved using the Laplace Transform and its properties, and the same answers are obtained.

16. *Convolution and the Laplace transform.*

   (a) Evaluate $h(t) = e^{-t} * e^{-2t}$ using direct itegration. (These signals are not defined for $t < 0$.)

   (b) Find H, the Laplace transform of h, using the expression for h you found in part (a).

   (c) Verify that H is the product of the Laplace transforms of e^{-t} and e^{-2t}.

   Solution:

   (a) To find $h(t)$ by using the definition of convolution:

$$
\begin{aligned}
e^{-t} * e^{-2t} &= \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau \\
&= e^{-2t} \int_0^t e^{\tau} d\tau \\
&= e^{-2t}(e^t - 1) \\
&= e^{-t} - e^{-2t}
\end{aligned}
$$

   (b) To find H, the Laplace Transform of h, use linearity and the Laplace Transform of the exponential:
$$
\mathcal{L}(e^{at}) = \frac{1}{s-a}
$$

   Hence,
$$
\mathcal{L}(e^{-t} - e^{-2t}) = \frac{1}{s+1} - \frac{1}{s+2}
$$

   (c) For this part, we are verifying that convolution in the time domain corresponds to multiplication in the s-domain. In other words, in part (a) and (b) we convolved two signals, and then took the Laplace transform of the resulting signal. We want to show this is the same thing as taking the Laplace Transform of each signal, and then multiplying them.
$$
(\mathcal{L}(e^{-t}))(\mathcal{L}(e^{-2t})) = (\frac{1}{s+1})(\frac{1}{s+2})
$$

   Which is in fact equal to the answer in part (b). These signals are sketched below.
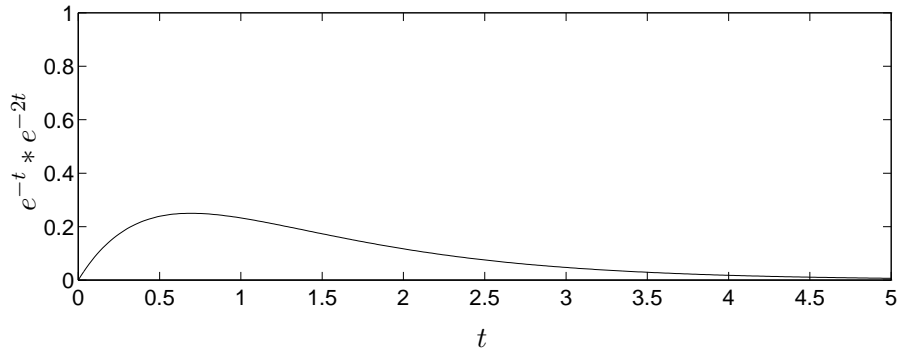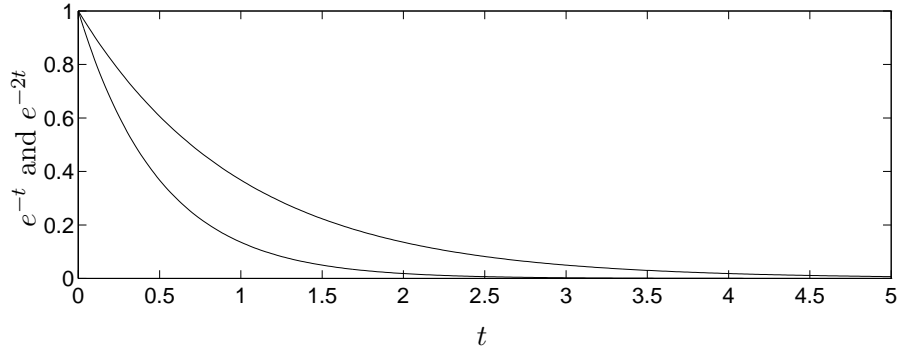
18. Consider the system shown below.
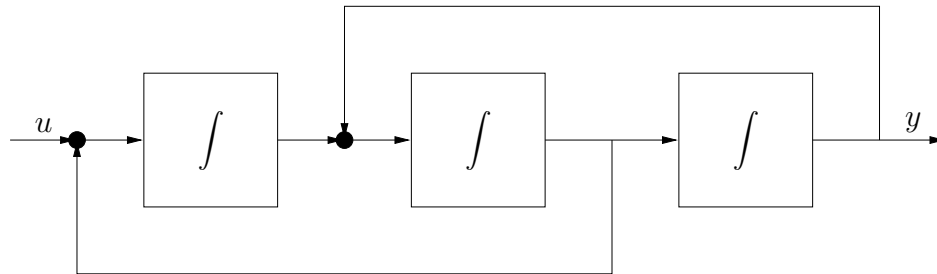
(a) Express the relation between u and y as an LCCODE. *Hint:* if the signal z is the output of an integrator, then its input is the signal z'.

(b) Assuming that $y(0) = y'(0) = y''(0) = 0$, derive an expression for Y (the Laplace transform of y) in terms of U (the Laplace transform of u).

Solution:

(a) The idea is to express the input to each integrator as the derivative of its output signal. The input to the 3rd (righthand) integrator is therefore y', and the input to the 2nd integrator is y''. The the output of the 1st integrator is $y'' - y$ (since the output plus y is equal to y''). The input of the 1st integrator is $u + y'$, which is also the derivative of $y'' - y$: $(y'' - y)' = u + y'$. This can be re-arranged as the LCCODE $y''' - 2y' = u$.

(b) Because the initial conditions are all zero, the Laplace transform of y''' is just $s^3 Y(s)$, and the Laplace transform of y' is $sY(s)$. Hence the Laplace transform of the LCCODE

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above is $s^3Y(s) - 2sY(s) = U(s)$. Solve for $Y(s)$ to get

$$Y(s) = \frac{U(s)}{s^3 - 2s}$$

19. Solve the following LCCODEs using Laplace transforms. Verify that the solution you find satisfies the initial conditions and the differential equation.

(a) $dv/dt = -2v + 3$, $v(0) = -1$.

(b) $d^2i/dt^2 + 9i = 0$, $i(0) = 1$, $di/dt(0) = 0$.

Solution:
We will use the differentiation theorem to take the Laplace transforms. Note that this step reduces differential equations in time to algebraic equations in the s-domain:

(a)

$$
\begin{aligned}
dv/dt &= -2v + 3 \\
sV(s) - v(0) &= -2V(s) + \frac{3}{s} \\
V(s) &= \frac{3 - s}{(s)(s + 2)} \\
&= \frac{3/2}{s} - \frac{5/2}{s + 2}
\end{aligned}
$$

thus $v(t) = \frac{3}{2} - \frac{5}{2}e^{-2t}$.

(b)

$$
\begin{aligned}
d^2i/dt^2 + 9i &= 0 \\
s^2I(s) - si(0) - \frac{di}{dt}(0) + 9I(s) &= 0 \\
I(s) &= \frac{s}{s^2 + 3^2}
\end{aligned}
$$

thus $i(t) = \cos 3t$.

20. Four signals a, b, c, and d are related by the differential equations

$$a' + a = b, \qquad b' + b = c, \qquad c' + c = d,$$

where $a(0) = b(0) = c(0) = 0$. Express $A(s)$, the Laplace transform of a, in terms of $D(s)$, the Laplace transform of d. **Solution.** There are several ways to solve this.
<u>First Method</u>
Starting with the equation for d, we have

$$
\begin{aligned}
d &= c' + c \\
&= b'' + b' + b' + b \\
&= a''' + a'' + 2a'' + 2a' + a' + a \\
&= a''' + 3a'' + 3a' + a \tag{1}
\end{aligned}
$$

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Now, $b'(0) = c(0) - b(0) = 0$, $a'(0) = b(0) - a(0) = 0$, and, consequently, $a''(0) = b'(0) - a'(0) = 0$. So, all of the initial conditions that we need to worry about (*i.e.*, $a''(0), a'(0), a(0)$) are zero. Then, taking the Laplace transform of both sides of (1), we obtain

$$D(s) = s^3 A(s) + 3s^2 A(s) + 3sA(s) + A(s)$$

Solving for A(s) produces

$$A(s) = \frac{D(s)}{s^3 + 3s^2 + 3s + 1}$$

<u>Second Method</u>
Since $a(0) = b(0) = c(0) = 0$, taking the Laplace transform of the three given equations produces

$$sA(s) + A(s) = B(s), \qquad sB(s) + B(s) = C(s), \qquad sC(s) + C(s) = D(s) \qquad (2)$$

Then, starting with the Laplace transform for $D(s)$, we obtain

$$
\begin{aligned}
D(s) &= sC(s) + C(s) \\
&= s^2 B(s) + sB(s) + sB(s) + B(s) \\
&= s^3 A(s) + s^2 A(s) + 2s^2 A(s) + 2sA(s) + sA(s) + A(s) \\
&= s^3 A(s) + 3s^2 A(s) + 3sA(s) + A(s)
\end{aligned}
$$

So, as before,

$$A(s) = \frac{D(s)}{s^3 + 3s^2 + 3s + 1}$$

<u>Yet Another Method</u>
From the equations (2), we first write them as

$$(s+1)A(s) = B(s), \qquad (s+1)B(s) = C(s), \qquad (s+1)C(s) = D(s),$$
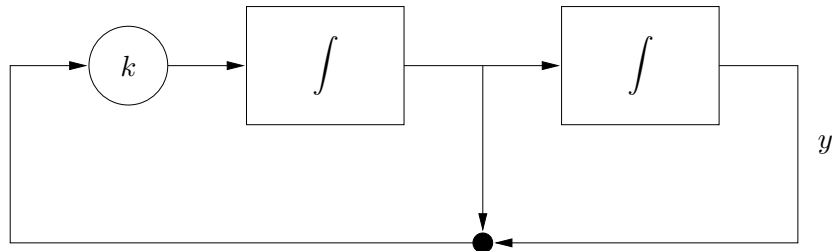
and now we express this as

$$A(s) = \frac{B(s)}{s+1}, \qquad B(s) = \frac{C(s)}{s+1}, \qquad C(s) = \frac{D(s)}{s+1}.$$

Now, combining these, we get

$$A(s) = \frac{1}{s+1}B(s) = \frac{1}{(s+1)^2}C(s) = \frac{1}{(s+1)^3}D(s),$$

which is the same answer.

23. In the system shown below, k is a gain.

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(a) For what values of k is this system stable?

(b) For what values of k is this system stable and critically damped?

(c) For what values of k do (nonzero) solutions y change sign infinitely often? (We *do not* require stability here.)

Solution:

The first thing to do is find the differential equation that describes the system. The signal entering the righthand integrator is y', and the signal entering the lefthand integrator is therefore y''. We therefore have $y'' = k(y' + y)$, *i.e.*,

$$y'' - ky' - ky = 0.$$

Now we can answer the questions.

(a) A second order LCCODE is stable when all its coefficients are positive. So for stability we require $k < 0$.

(b) Critically damped means the roots of the characteristic polynomial are equal, *i.e.*, $(-k)^2 + 4k = 0$. Thus $k = 0$ or $k = -4$. We also require stability so the only possible solution is $k = -4$.

(c) The first thing to do is to decode what is being asked. If a signal is a sum of two exponentials, it can change sign at most once. So if the signal change sign infinitely often, it means the solution is an exponentially growing or decaying sinusoid; the characteristic polynomial has complex roots. This means that $(-k)^2 + 4k < 0$. We write this as $k(k + 4) < 0$. To figure out which k's satisfy this inequality, note that if $k \leq -4$, then $k(k + 4) \geq 0$. In a similar way if $k \geq 0$, $k(k + 4) \geq 0$. So what we need is $-4 < k < 0$. This one wasn't so easy!

26. *Thermal runaway.* A conductor with resistance R carries a fixed positive current i, and hence dissipates a power $P = i^2 R$. This causes the conductor to heat up (hopefully not too much) above the ambient temperature.

Let $T(t)$ denote the temperature of the conductor above the ambient temperature at time t. T satisfies the equation
$$aT' = -bT + P$$

where $a > 0$ is the thermal capacity of the conductor (in J/°C), $b > 0$ is the thermal conductivity (in W/°C), and P is the power (in W) dissipated in the conductor.

The resistance R of the conductor changes with temperature according to

$$R = R_0(1 + cT)$$

where the constant c (which has units 1/°C) is called the *resistance temperature coefficient* (or just 'tempco') of the conductor, and $R_0 > 0$ is the resistance of the conductor at ambient temperature. Depending on the material of the conductor, the tempco c can be positive or negative; for example, for metal wires the tempco is positive. (The formula above is valid only over a range where $1 + cT > 0$.)

(a) Consider a metal wire, for which $c > 0$. If the current i is smaller than a critical value i_{crit}, the temperature T converges to a steady-state value as $t \to \infty$. If the current i is larger than this critical value of current, then the temperature T converges to ∞ as $t \to \infty$. (In practice, the temperature increases until the conductor is destroyed, e.g., melted.) This phenomenon is called *thermal runaway.*

Find the critical value i_{crit}, above which thermal runaway occurs. Express the answer in terms of the other constants in the problem (a, b, R_0, c).

(b) Suppose the wire is initially at ambient temperature, i.e., $T(0) = 0$, and the constants have the values

$$a = 1\text{J}/^\circ\text{C}, \quad b = 0.5\text{W}/^\circ\text{C}, \quad i = 10\text{A}, \quad R_0 = 1\Omega, \quad c = 0.01/^\circ\text{C}$$

Find $T(t)$ for $t \geq 0$

Solution:

(a) We just combine the two given equations to get

$$aT' = -bT + i^2 R_0 (1 + cT),$$

which we can write as

$$T' = \frac{-b + i^2 R_0 c}{a} T + \frac{i^2 R_0}{a}.$$

This simple first order equation is stable only if the coefficient $(-b + i^2 R_0 c)/a$ is negative, i.e., only if

$$|i| \leq \sqrt{b/(R_0 c)} = i_{\text{crit}}.$$

You can solve the ODE to verify that if $|i| \geq i_{\text{crit}}$, the temperature goes to ∞; for $|i| < i_{\text{crit}}$, the temperature converges to some steady-state value.

(b) Plugging in the values given, our differential equation becomes $T' = 0.5T + 100$. We can solve this via Laplace transforms. Let Y denote the Laplace transform of T (which, unfortunately, is already capitalized!):

$$sY = 0.5Y + 100/s,$$

so

$$Y = \frac{100}{s(s - 0.5)} = \frac{-200}{s} + \frac{200}{s - 0.5}.$$

Therefore we have

$$y(t) = 200\left(e^{0.5t} - 1\right).$$

Note that in this case i is above the critical current; we have thermal runaway here.

27. A voltage $v(t)$ is applied to a DC motor. A simple electrical model of the motor is an inductance L in series with a resistance R, so the motor current $i(t)$ satisfies

$$L di/dt + Ri = v.$$

The motor shaft angle is denoted $\theta(t)$, and the shaft angular velocity $\omega(t)$ (so we have $\omega = d\theta/dt$). The motor current puts a torque on the shaft equal to $ki(t)$, where k is the *motor*

constant. The shaft rotational inertia is J and the damping coefficient is b. Newton's equation is then:

$$Jd\omega/dt = ki - b\omega.$$

Assuming that $i(0) = 0$, $\theta(0) = 0$, and $\omega(0) = 0$, express Θ (the Laplace transform of θ) in terms of V (the Laplace transform of v).

The numbers L, R, k, J, b are all positive constants.

Solution:
Everything you need to know is in the three equations

$$Ldi/dt + Ri = v, \quad \omega = d\theta/dt, \quad Jd\omega/dt = ki - b\omega.$$

Let's take the Laplace transform of these equations to get the algebraic equations

$$L(sI - i(0)) + RI = V, \quad \Omega = s\Theta - \theta(0), \quad J(s\Omega - \omega(0)) = kI - b\Omega.$$

From the assumption that $i(0) = 0$, $\theta(0) = 0$, and $\omega(0) = 0$, these equations simplify to

$$LsI + RI = V, \quad \Omega = s\Theta, \quad Js\Omega = kI - b\Omega.$$

Now we solve these equations for Θ in terms of V (and the constants). From the first equation we get

$$I = \frac{V}{Ls + R}.$$

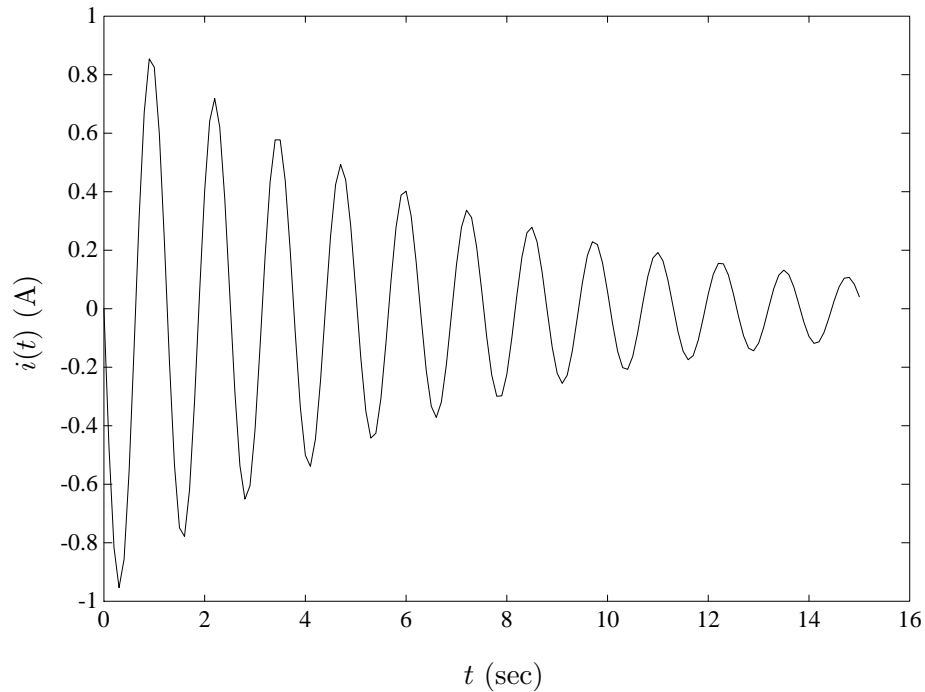From the third equation we get

$$\Omega = \frac{kI}{Js + b}.$$

Combining these we get

$$\Omega = \frac{kV}{(Js + b)(Ls + R)}.$$

Combining this with the middle equation above yields

$$\Theta = \frac{kV}{s(Js + b)(Ls + R)}.$$

30. The waveform shown below is the current in a series RLC circuit. The value of the resistor is 100Ω.

t (sec)

(a) Estimate L and C.

(b) About how long will it be before 99% of the initial stored energy in the circuit has dissipated?

Solution:

(a) The waveform is obviously one of an underdamped circuit. From the notes, we know that $i(t) = Ae^{-\alpha t}\cos(\omega_\mathrm{d}t + \phi)$, with $\alpha = R/2L$ and $\omega_\mathrm{d} = \sqrt{1/LC - \alpha^2}$. We can estimate α and ω_d directly from the waveform. If we look at its amplitude, we see that it decreases from 1 to 0.2 in approximately 11.6s. This yields the equation $e^{-11.6\alpha} \approx 0.2$, that is $\alpha \approx 0.14$. Since $L = R/2\alpha$, we deduce $L = 360$H. Evaluating ω_d can be done by counting the cycles on the graph. For example, 9.5 cycles have elapsed in approximately 12s, therefore the period is about 1.25s and thus $\omega_\mathrm{d} \approx 5$rad/s. Using the equation for ω_d, we have $C = 1/L(\omega_\mathrm{d}^2 + \alpha^2)$, or, numerically, $C \approx 1.1 \cdot 10^{-4}$F $= 0.11$mF.

(b) The energy in the circuit is proportional to $i(t)^2$, that is, it decays exponentially with a rate of 2α. Therefore, it will be approximately one percent of its original value after a time T such that $2\alpha T \approx 5$, or $T \approx 18$s.

37. For each of the following rational functions, find the poles and zeros (giving multiplicities of each), the real factored form, the partial fraction expansion, and inverse Laplace transform. (In some cases, the expression may already be in one of these forms.)

(a) $\dfrac{1}{s+1} + \dfrac{1}{s+2} + \dfrac{1}{s+3}$

(b) $\dfrac{s^2 + 1}{s^3 - s}$

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(c) $\dfrac{(s-2)(s-3)(s-4)}{s^4-1}$

Solution:

(a) This one is already in partial fraction form. We can see immediately the poles are -1, -2, and -3. From the partial fraction expansion we can just read off the inverse Laplace transform: $e^{-t} + e^{-2t} + e^{-3t}$.

To find the zeros and real factored form we'll put everything on a common denominator:

$$\frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3} = \frac{(s+2)(s+3) + (s+1)(s+3) + (s+1)(s+2)}{(s+1)(s+2)(s+3)}$$

$$= \frac{3s^2 + 12s + 11}{(s+1)(s+2)(s+3)}$$

To find the zeros we apply the quadratic formula to the numerator. This yields zeros at $s = -2 \pm \frac{\sqrt{3}}{3}$. These zeros are complex, so the last expression above is the real factored form.

(b) We cannot factor the numerator over the reals (*i.e.*, it has complex roots), and the denominator can be written as $s(s+1)(s-1)$. This gives us the real factored form:

$$\frac{s^2+1}{s(s+1)(s-1)}.$$

The zeros are $\pm j$, and the poles are 0, ± 1:

$$\frac{s^2+1}{s^3-s} = \frac{(s+j)(s-j)}{s(s+1)(s-1)}$$

Performing the partial fraction expansion, we find

$$\frac{s^2+1}{s^3-s} = \frac{-1}{s} + \frac{1}{s-1} + \frac{1}{s+1}$$

Thus the inverse Laplace transform is $-1 + e^t + e^{-t}$.

(c) The numerator is already in real factored form; the zeros are 2, 3, and 4. Now let's factor the denominator. In fact there is an explicit but very ugly formula that gives all four roots of a general quartic (just like the quadratic formula), but we won't need it here. (And by the way, there is no such formula for the roots of polynomials of degree five and higher!) To factor it we just notice that 1 and -1 are roots, or notice that we can write s^4-1 as $(s^2-1)(s^2+1)$, and we can factor both terms. Either way we end up with $s^4 - 1 = (s+1)(s-1)(s^2+1)$. This is as far as we can go over the reals. So the real factored form is

$$\frac{(s-2)(s-3)(s-4)}{(s+1)(s-1)(s^2+1)}.$$

The denominator factors further to $(s+1)(s-1)(s+j)(s-j)$. Thus the poles are ± 1 and $\pm j$. The partial fraction expansion is

$$\frac{15}{s+1} - \frac{1.5}{s-1} + \frac{-6.25 + 3.75j}{s+j} + \frac{-6.25 - 3.75j}{s-j}$$

hence the inverse Laplace transform is

$$15e^{-t} - 1.5e^t - 12.5\cos t + 7.5\sin t.$$