Homework 1 solutions

1. *Optimizing gains in a two-stage amplifier.* Consider the two-stage amplifier described on page 2-12 of the lecture notes. In this problem you will determine optimal values for the two amplifier gains a_1 and a_2. The constraints and specifications are:

   - The amplifier gains can be varied from 5 (14dB) to 20 (26dB).
   - Each of the noises is a voltage with a magnitude no more than 100μV, *i.e.*, $|n_1| \leq 10^{-4}$, $|n_2| \leq 10^{-4}$.
   - The input signal voltage ranges between $\pm$100mV.
   - The gain from the input signal to the output signal must be 100, *i.e.*, if the noises were zero we would have $y = 100u$.
   - The maximum allowed voltage magnitude is 1V at the output of the first amplifier, and 10V at the output of the second amplifier. (Effects of the noise voltages can be ignored in this calculation.)

   Find choices for a_1 and a_2 that satisfy the specifications while minimizing the largest possible effect of the noise voltages at the output. Explain what you are doing, and what your reasonning is.

   Solution:
   From the specifications, it follows that $a_1a_2 = 100$. Combining this information with the equation at the bottom of page 2-12, it follows that $y = 100u + 100n_1 + a_2n_2$. This shows that the amount of noise from n_1 is fixed; it will range between $\pm 100 \cdot 100\mu$V, *i.e.*, $\pm$10mV. But we *can* minimize the effects of the noise n_2, since it is multiplied by a_2. This suggests we should make a_2 as small as can, *i.e.*, 5 (which implies $a_1 = 20$). Let's see if this initial design satisfies the other specifications. With $a_1 = 20$, then the output of the first stage is $\pm$2V, which is twice the maximum allowed. So we must reduce the gain of the first stage by a factor of two in order to not violate this constraint. This yields our second design:

   $$a_1 = a_2 = 10.$$

   Now we can check that all the specs are satisfied. The output of the second amplifier ranges between $\pm$10V, which is just at its limit. The total gain is 100, and the effect of the noises is minimized. The maximum possible effect of the noises on the output is $\pm$10mV from n_1 and $\pm$1mV from n_2; all together the noises can cause the output to vary between $\pm$11mV.

   The *dynamic range* of an amplifier is the ratio of the largest possible output signal swing to the largest possible output noise swing. For this amplifier the dynamic range is 10V/11mV = 909, or 59dB.

   Although this problem is really simple, it gives the idea of real problems that are encountered in amplifier design.

2. *Small signals.* In lecture 1 we mentioned several methods for determining the size of a signal. Intuition suggests that even though they are not the same, the measures shouldn't be too

different. After all, a small signal is a small signal, right? In this problem we explore this issue.

Consider a family of signals described by

$$u(t) = \begin{cases} 1/\sqrt{d}, & 0 \leq t \leq d \\ 0, & d < t < 1 \end{cases}$$

for $0 \leq t < 1$, and periodic with period 1 (*i.e.*, the signal repeats every second). The parameter d, which satisfies $0 < d < 1$, is called the *duty cycle* of the periodic pulse signal.

Sketch the signal for a few values of d. What is its peak, RMS, and average-absolute value? As the duty-cycle d approaches 0, is the signal getting smaller or larger?

Solution:
From the formulas on page 1-7, the peak value is $\frac{1}{\sqrt{d}}$, the RMS value is 1 and the average-absolute value is $\sqrt{d}$. As $d \to 0$, the peak value $\to \infty$, the RMS value remains fixed at 1, and the average-absolute value $\to 0$.

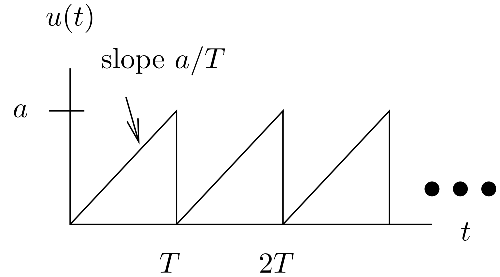
In particular for very small d, the peak is very large, the RMS value is one, and the average-absolute is very small. In other words, depending on what method you use to measure the 'size' of this signal, you can get answers ranging from 'very small' to 'very large'.

You might find this really confusing, but don't worry. In any practical application there will be some most appropriate method to measure the signal size. Also, for most (but obviously not all) signals, these measures give answers that aren't too far off. (Sinusoids give a good example; see the lecture notes.)

3. A *sawtooth signal* u has the form $u(t) = at/T$ for $0 \leq t < T$, and is T-periodic (*i.e.*, repeats every T seconds). The constant a is called the *amplitude* of the signal, and the constant T (which is positive) is called the *period* of the signal. You can assume that $a > 0$.

   (a) Find the peak value of a sawtooth signal.
   (b) Find the RMS (root-mean-square) value of a sawtooth signal.
   (c) Find the AA (average-absolute) value of a sawtooth signal.
   (d) In the space below, sketch the derivative of a sawtooth signal. Be sure to label all axes, slopes, magnitudes of any impulses, etc.

   Solution:
   Let's start by plotting the sawtooth function:



   (a) The peak is the maximum of the absolute value of the signal, which clearly is a.

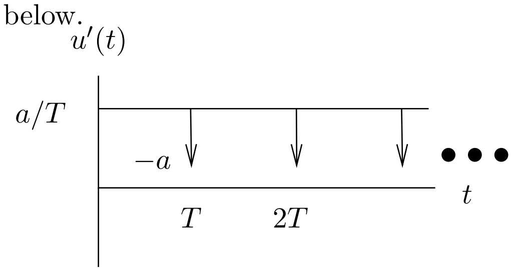
(b) Since the signal is periodic we can as well integrate over just one cycle. The RMS value of the signal is given by

$$\begin{aligned} \mathrm{RMS}(u) &= \left(\frac{1}{T}\int_0^T u(t)^2\,dt\right)^{1/2} \\ &= \left(\frac{1}{T}\int_0^T (at/T)^2\,dt\right)^{1/2} \\ &= a/\sqrt{3} \end{aligned}$$

(c) Here too we can integrate over one cycle.

$$\mathrm{AA}(u) = \frac{1}{T}\int_0^T |u(t)|\,dt = \frac{1}{T}\int_0^T (at/T)\,dt = a/2.$$

(d) Whenever t is not a multiple of T, the sawtooth signal is differentiable, with constant derivative (slope) a/T. At every multiple of T, the sawtooth signal jumps down by a. Therefore at every multiple of T, the derivative includes an impulse that has magnitude $-a$. This is sketched below.



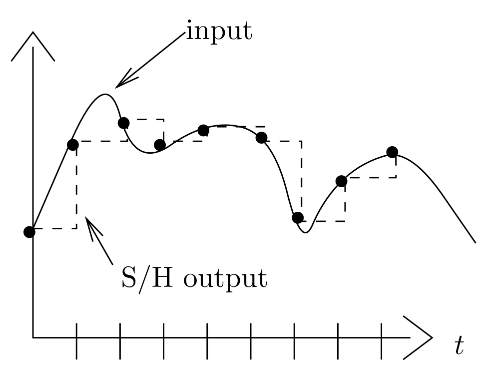
4. *Sample and hold system.* A sample and hold (S/H) system, with sample time h, is described by $y(t) = u(h\lfloor t/h \rfloor)$, where $\lfloor a \rfloor$ denotes the largest integer that is less than or equal to a.

Sketch an input and corresponding output signal for a S/H, to illustrate that you understand what it does.

Is a S/H system linear?

Solution:
First let's dicsuss the complicated notation we used. $\lfloor t/h \rfloor$ gives the number of sample times that have occured at time t. Multiplying by h, which yields $h\lfloor t/h \rfloor$ gives the time of the last sample time. The output of a S/H at time t is given by the input signal value at the last sample time, *i.e.*, $u(h\lfloor t/h \rfloor)$.

A sketch of an input signal and its corresponding output is shown below:



The S/H system *is* linear. To check this we must check the homogeneity property and the superposition property. For superposition, suppose you have two signals, u and v, and consider the signal $u + v$. For any t we have

$$(u + v)(h\lfloor t/h \rfloor) = u(h\lfloor t/h \rfloor) + v(h\lfloor t/h \rfloor)$$

which shows that superposition holds. A similar argument shows that a S/H is homogenous.