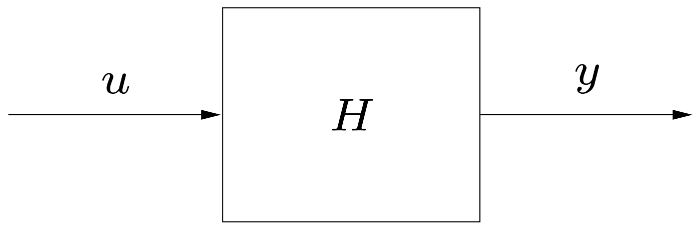
Lecture 10
Sinusoidal steady-state and frequency response

- sinusoidal steady-state
- frequency response
- Bode plots

Response to sinusoidal input

convolution system with impulse response h, transfer function H



sinusoidal input $u(t) = \cos(\omega t) = \left(e^{j\omega t} + e^{-j\omega t}\right)/2$

output is $y(t) = \int_0^t h(\tau)\cos(\omega(t-\tau))\, d\tau$

let's write this as

$$y(t) = \int_0^\infty h(\tau)\cos(\omega(t-\tau))\, d\tau - \int_t^\infty h(\tau)\cos(\omega(t-\tau))\, d\tau$$

- first term is called *sinusoidal steady-state* response
- second term decays with t if system is stable; if it decays it is called the *transient*

if system is stable, sinusoidal steady-state response can be expressed as

$$\begin{aligned} y_{\mathrm{sss}}(t) &= \int_0^\infty h(\tau)\cos(\omega(t-\tau))\,d\tau \\ &= (1/2)\int_0^\infty h(\tau)\left(e^{j\omega(t-\tau)}+e^{-j\omega(t-\tau)}\right)\,d\tau \\ &= (1/2)e^{j\omega t}\int_0^\infty h(\tau)e^{-j\omega\tau}\,d\tau + (1/2)e^{-j\omega t}\int_0^\infty h(\tau)e^{j\omega\tau}\,d\tau \\ &= (1/2)e^{j\omega t}H(j\omega) + (1/2)e^{-j\omega t}H(-j\omega) \\ &= (\Re H(j\omega))\cos(\omega t) - (\Im H(j\omega))\sin(\omega t) \\ &= a\cos(\omega t+\phi) \end{aligned}$$

where $a = |H(j\omega)|$, $\phi = \angle H(j\omega)$

conclusion

if the convolution system is stable, the response to a sinusoidal input is asymptotically sinusoidal, with the same frequency as the input, and with magnitude & phase determined by $H(j\omega)$

- $|H(j\omega)|$ gives *amplification factor*, $i.e.$, RMS(y_{ss})/RMS(u)
- $\angle H(j\omega)$ gives *phase shift* between u and y_{ss}

special case: $u(t) = 1$ ($i.e.$, $\omega = 0$); output converges to $H(0)$ (DC gain)

frequency response

transfer function evaluated at $s = j\omega$, $i.e.$,

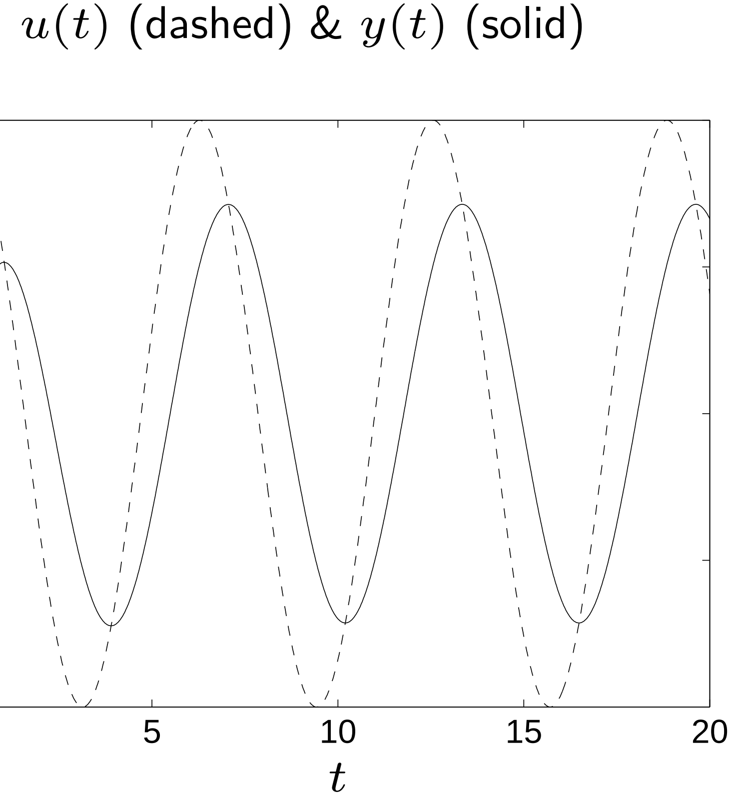
$$H(j\omega) = \int_0^\infty h(t)e^{-j\omega t}dt$$

is called *frequency response* of the system

since $H(-j\omega) = \overline{H(j\omega)}$, we usually only consider $\omega \geq 0$

Example

- transfer function $H(s) = 1/(s+1)$
- input $u(t) = \cos t$
- SSS output has magnitude $|H(j)| = 1/\sqrt{2}$, phase $\angle H(j) = -45^\circ$



more generally, if system is stable and the input is asymptotically sinusoidal, $i.e.$,

$$u(t) \to \Re\left(\mathbf{U}e^{j\omega t}\right)$$

as $t \to \infty$, then

$$y(t) \to y_{\rm ss}(t) = \Re\left(\mathbf{Y}_{\rm ss}e^{j\omega t}\right)$$

as $t \to \infty$, where

$$\mathbf{Y}_{\rm ss} = H(j\omega)\mathbf{U}$$

$$H(j\omega) = \frac{\mathbf{Y}_{\rm ss}}{\mathbf{U}}$$

for a stable system, $H(j\omega)$ gives ratio of phasors of asymptotic sinusoidal output & input thus, for example (assuming a stable system),

- $H(j\omega)$ large means asymptotic response of system to sinusoid with frequnecy ω is large

- $H(0) = 2$ means asymptotic response to a constant signal is twice the input value

- $H(j\omega)$ small for large ω means the asymptotic output for high frequency sinusoids is small

Measuring frequency response

for $\omega = \omega_1, \ldots, \omega_N$,

- apply sinusoid at frequency ω, with phasor $\mathbf{U}$
- wait for output to converge to SSS
- measure $\mathbf{Y}_{\mathrm{ss}}$
  ($i.e.$, magnitude and phase shift of y_{ss})

N can be a few tens (for hand measurements) to several thousand

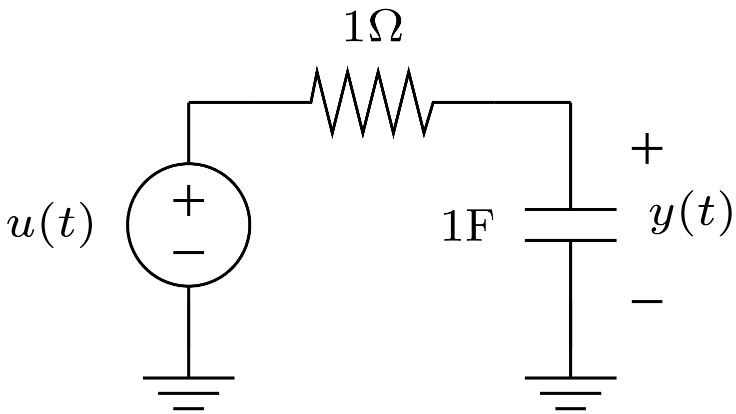
Frequency response plots

frequency response can be plotted in several ways, $e.g.$,

- $\Re H(j\omega)$ & $\Im H(j\omega)$ versus ω
- $H(j\omega) = \Re H(j\omega) + j\Im H(j\omega)$ as a curve in the complex plane (called *Nyquist plot*)
- $|H(j\omega)|$ & $\angle H(j\omega)$ versus ω (called *Bode plot*)

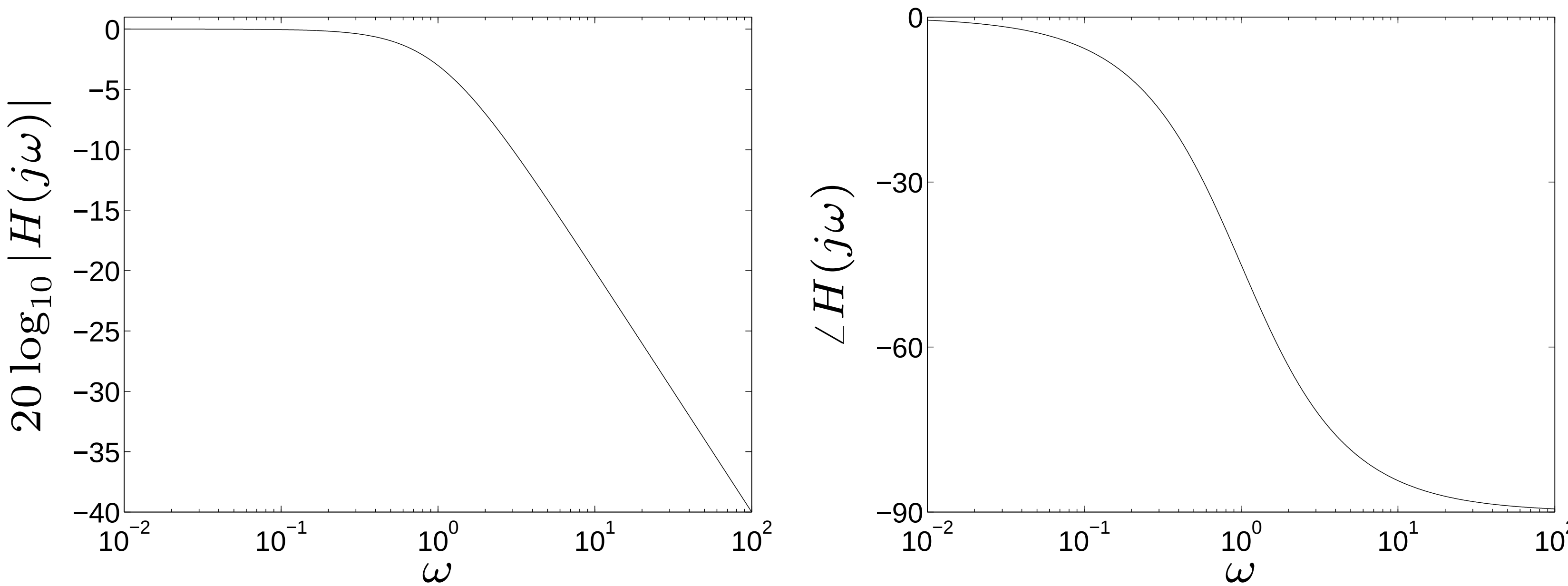
the most common format is a Bode plot

example: RC circuit



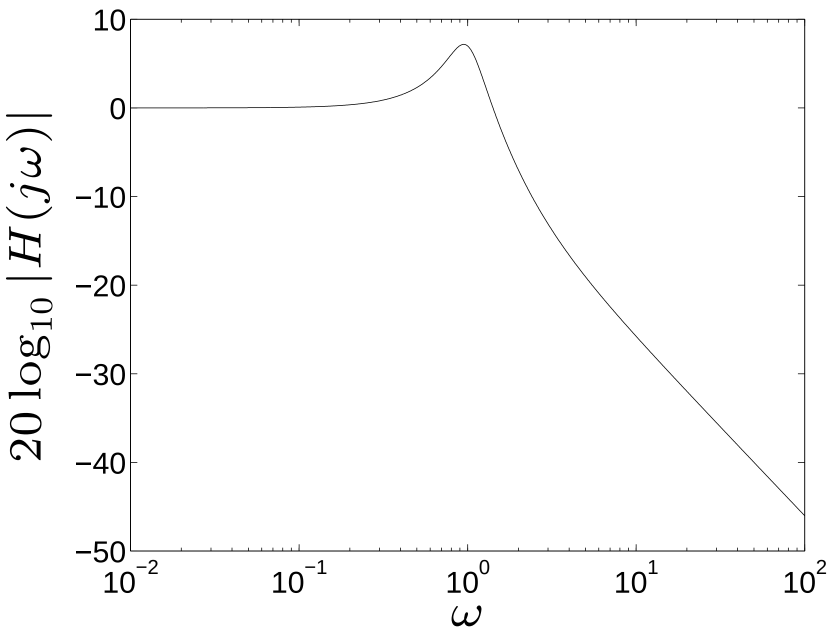
$$Y(s) = \frac{1}{1+s} U(s)$$

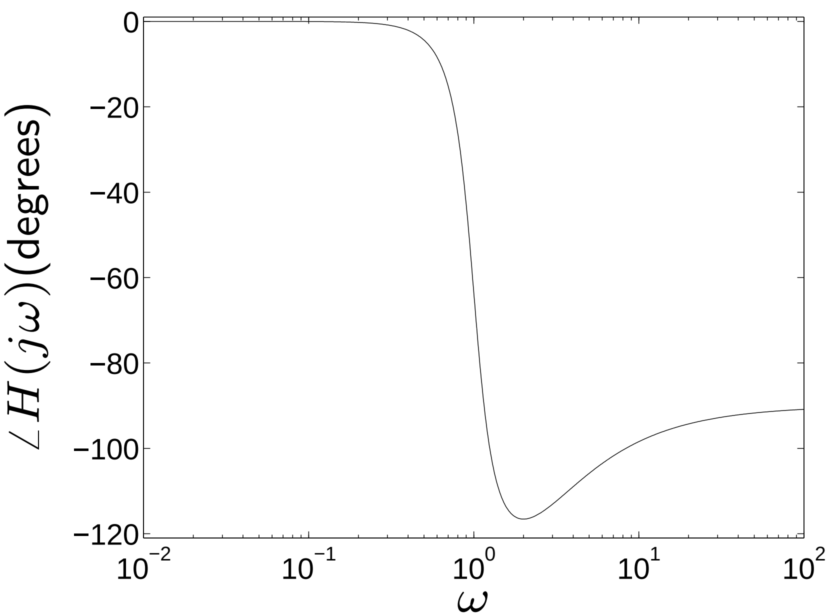
$$H(j\omega) = \frac{1}{1+j\omega}$$



example: suspension system of page 8-6 with $m = 1$, $b = 0.5$, $k = 1$,

$$H(s) = \frac{0.5s + 1}{s^2 + 0.5s + 1}, \qquad H(j\omega) = \frac{(-0.75\omega^2 + 1) - j0.5\omega^3}{\omega^4 - 1.75\omega^2 + 1}$$





Bode plots

frequency axis

- logarithmic scale for ω
- *horizontal* distance represents a fixed frequency ratio or factor: ratio $2:1$ is called an *octave*; ratio $10:1$ is called a *decade*

magnitude $|H(j\omega)|$

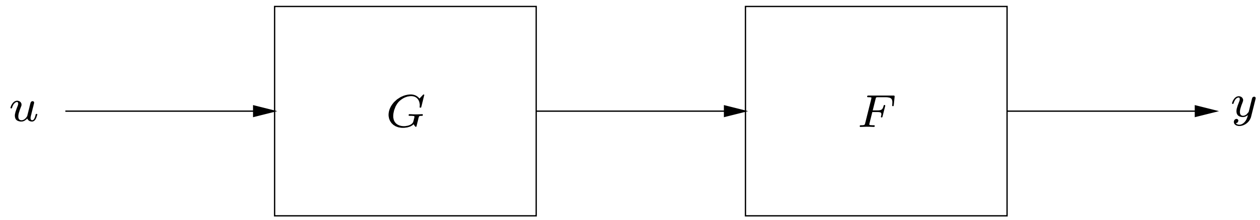
- expressed in dB, *i.e.*, $20\log_{10}|H(j\omega)|$
- *vertical* distance represents dB, *i.e.*, a fixed ratio of magnitudes ratio $2:1$ is $+6$dB, ratio $10:1$ is $+20$dB
- *slopes* are given in units such as dB/octave or dB/decade

phase $\angle H(j\omega)$

- multiples of 360° don't matter
- phase plot is called *wrapped* when phases are between $\pm180^\circ$ (or $0, 360^\circ$); it is called *unwrapped* if multiple of 360° is chosen to make phase plot continuous

Bode plots of products

consider product of transfer functions $H = FG$:



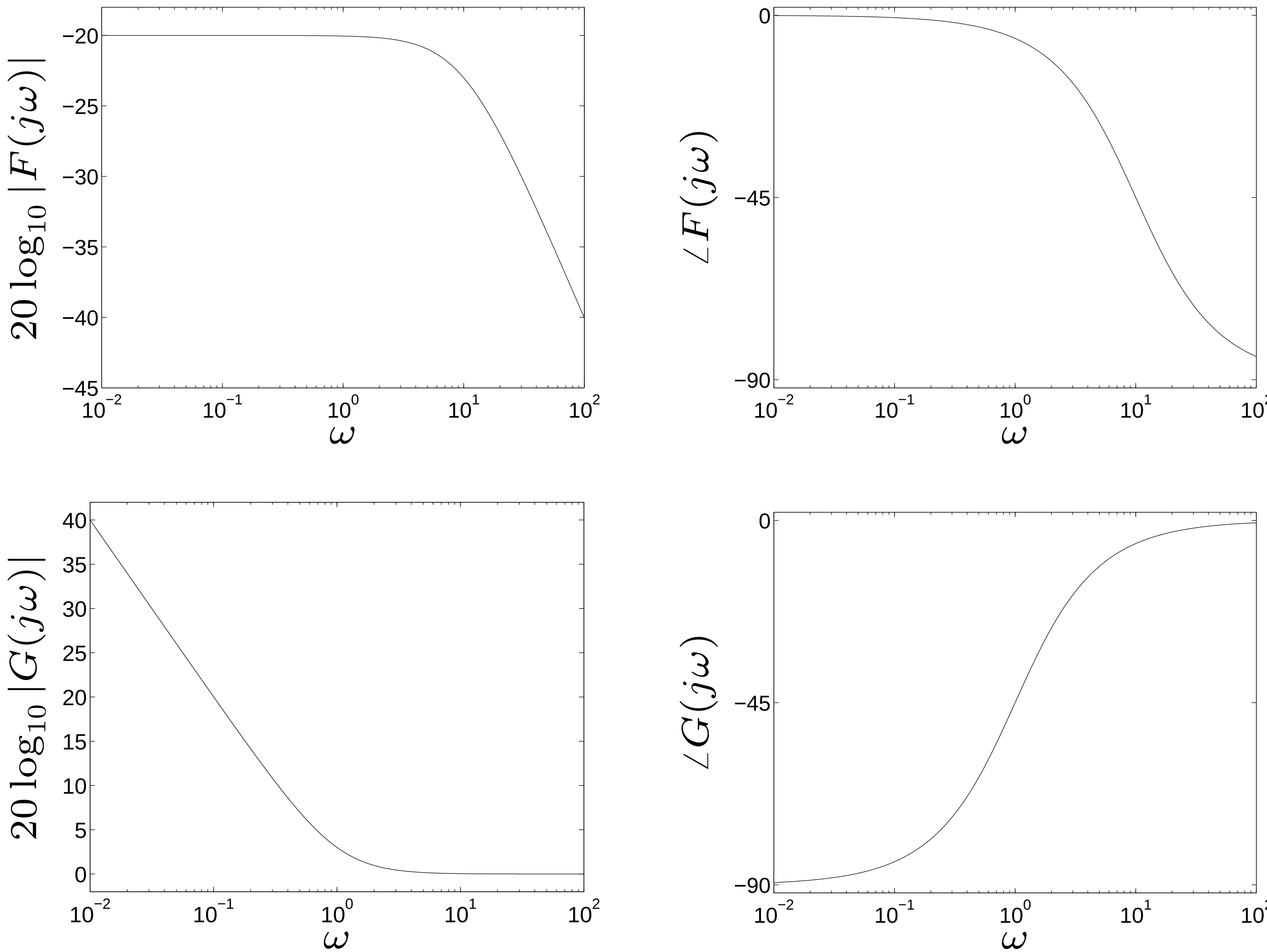
frequency response magnitude and phase are

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= 20\log_{10}|F(j\omega)| + 20\log_{10}|G(j\omega)| \\ \angle H(j\omega) &= \angle F(j\omega) + \angle G(j\omega) \end{aligned}$$

here we use the fact that for $a, b \in \mathbf{C}$, $\angle(ab) = \angle a + \angle b$

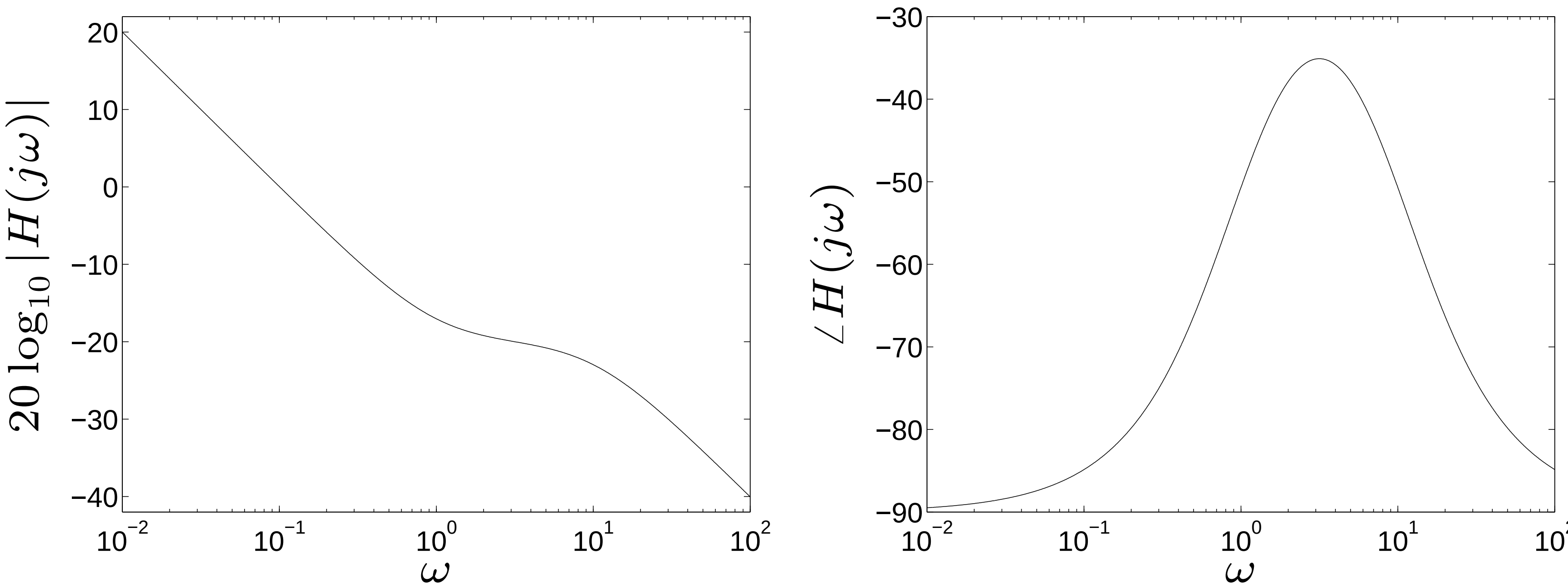
so, Bode plot of a product is the sum of the Bode plots of each term (extends to many terms)

example. $F(s) = 1/(s+10)$, $G(s) = 1 + 1/s$



Bode plot of

$$H(s) = F(s)G(s) = \frac{1 + 1/s}{s + 10} = \frac{s + 1}{s(s + 10)}$$



Bode plots from factored form

rational transfer function H in factored form:

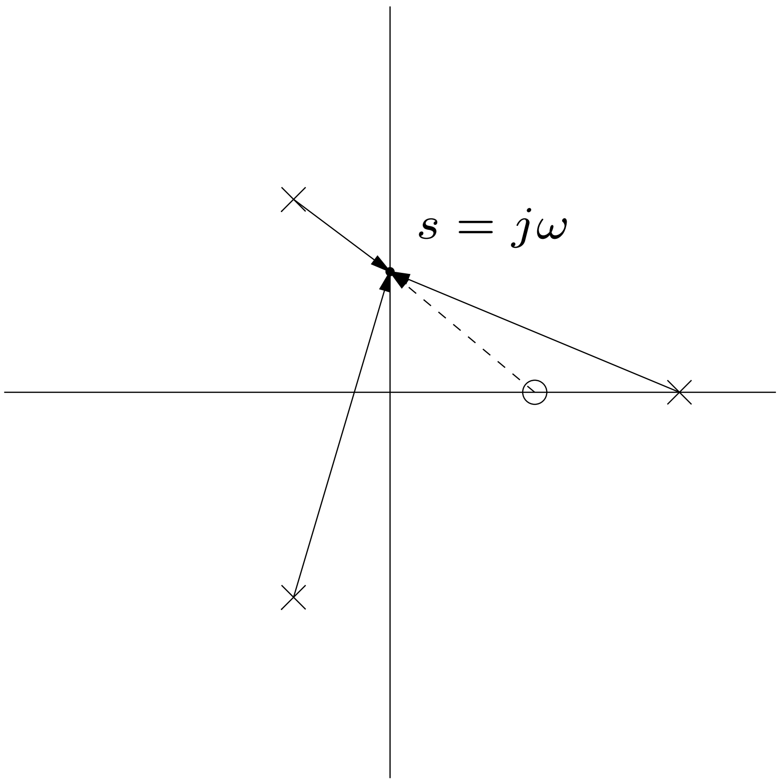
$$H(s) = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= 20\log_{10}|k| + \sum_{i=1}^{m} 20\log_{10}|j\omega - z_i| \\ &\quad - \sum_{i=1}^{n} 20\log_{10}|j\omega - p_i| \end{aligned}$$

$$\angle H(j\omega) = \angle k + \sum_{i=1}^{m} \angle(j\omega - z_i) - \sum_{i=1}^{n} \angle(j\omega - p_i)$$

(of course $\angle k = 0^\circ$ if $k > 0$, and $\angle k = 180^\circ$ if $k < 0$)

Graphical interpretation: $|H(j\omega)|$



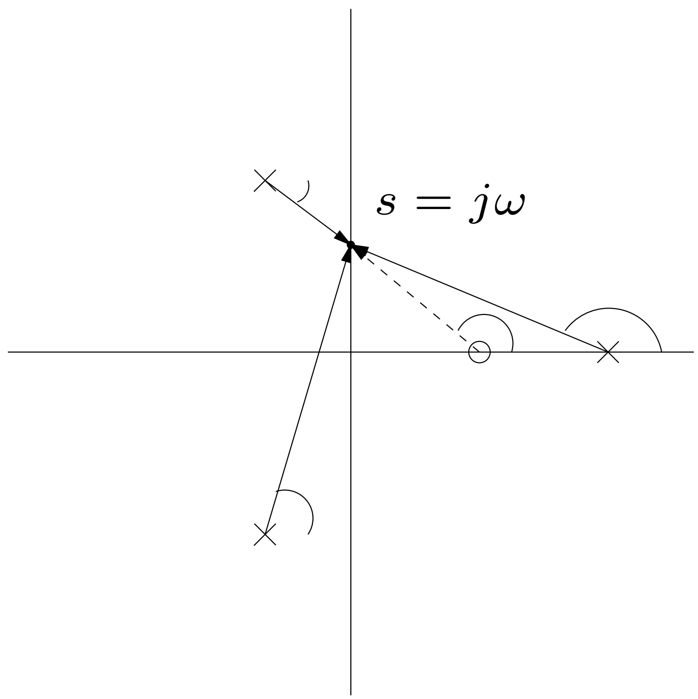
$$|H(j\omega)| = |k| \frac{\prod_{i=1}^{m} \mathsf{dist}(j\omega, z_i)}{\prod_{i=1}^{n} \mathsf{dist}(j\omega, p_i)}$$

since for $u, v \in \mathbf{C}$, $\mathsf{dist}(u, v) = |u - v|$

therefore, *e.g.*:

- $|H(j\omega)|$ gets big when $j\omega$ is near a pole
- $|H(j\omega)|$ gets small when $j\omega$ is near a zero

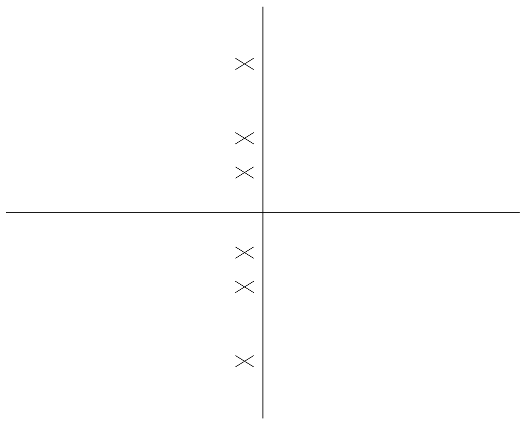
Graphical interpretation: $\angle H(j\omega)$



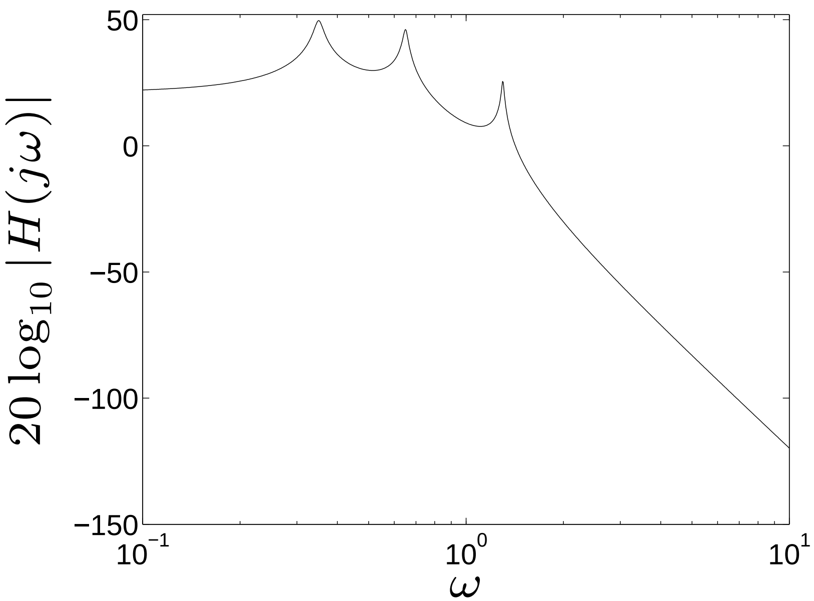
$$\angle H(j\omega) = \angle k + \sum_{i=1}^{m} \angle(j\omega - z_i) - \sum_{i=1}^{n} \angle(j\omega - p_i)$$

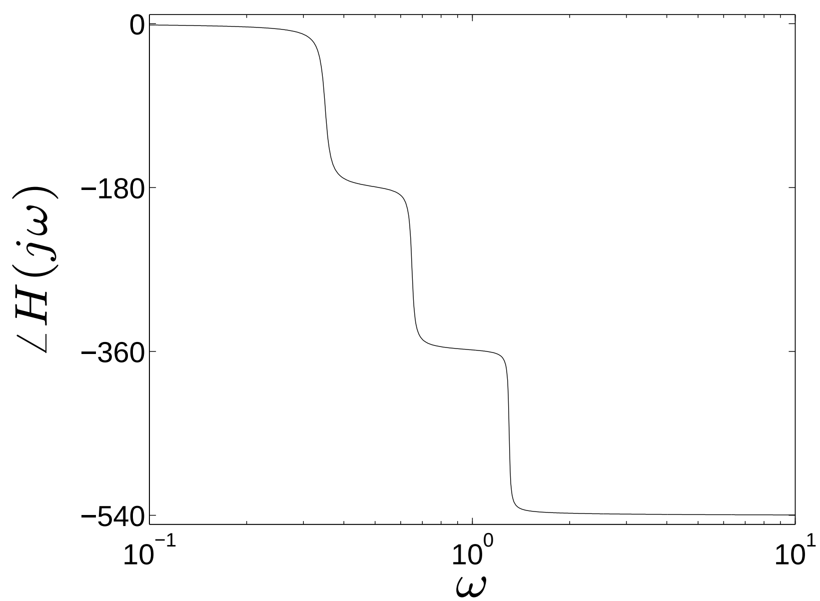
therefore, $\angle H(j\omega)$ changes rapidly when a pole or zero is near $j\omega$

Example with lightly damped poles

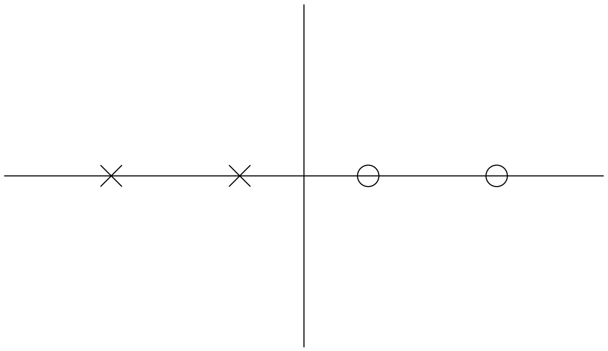


poles at $s = -0.01 \pm j0.2$, $s = -0.01 \pm j0.7$, $s = -0.01 \pm j1.3$,

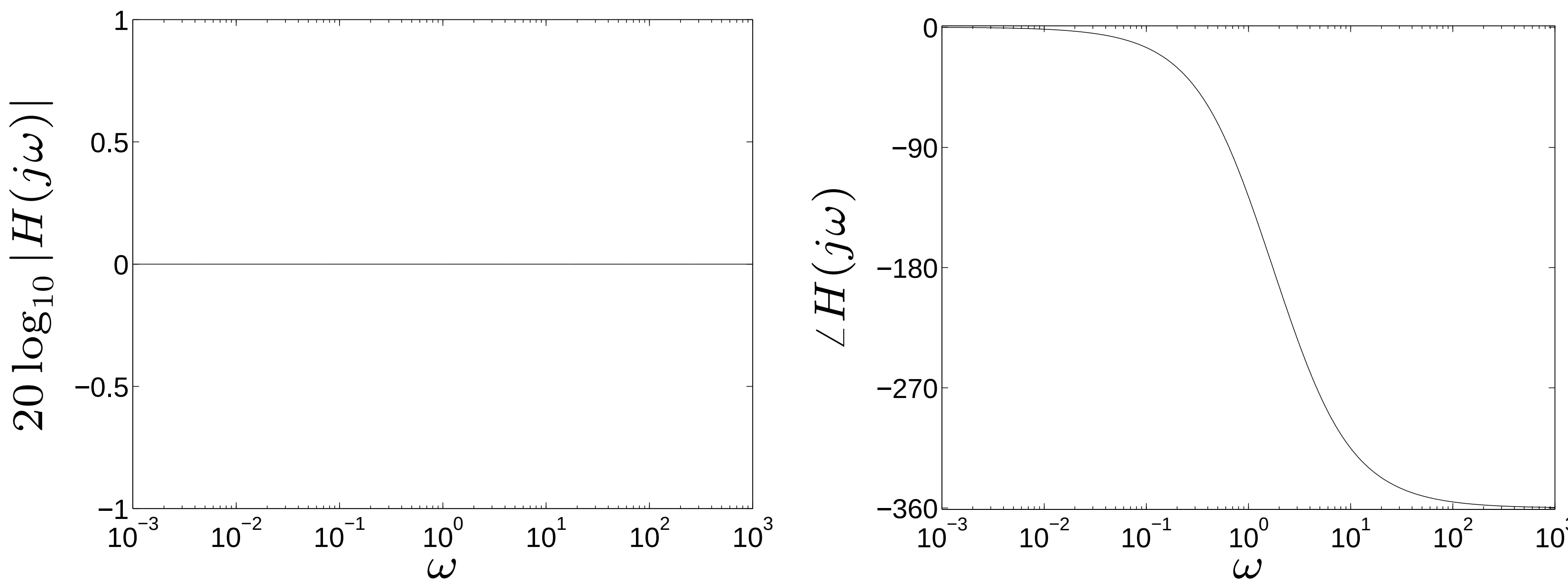




All-pass filter



$$H(s) = \frac{(s-1)(s-3)}{(s+1)(s+3)}$$



called *all-pass filter* since gain magnitude is one for all frequencies

Analog lowpass filters

analog lowpass filters: approximate ideal lowpass frequency response

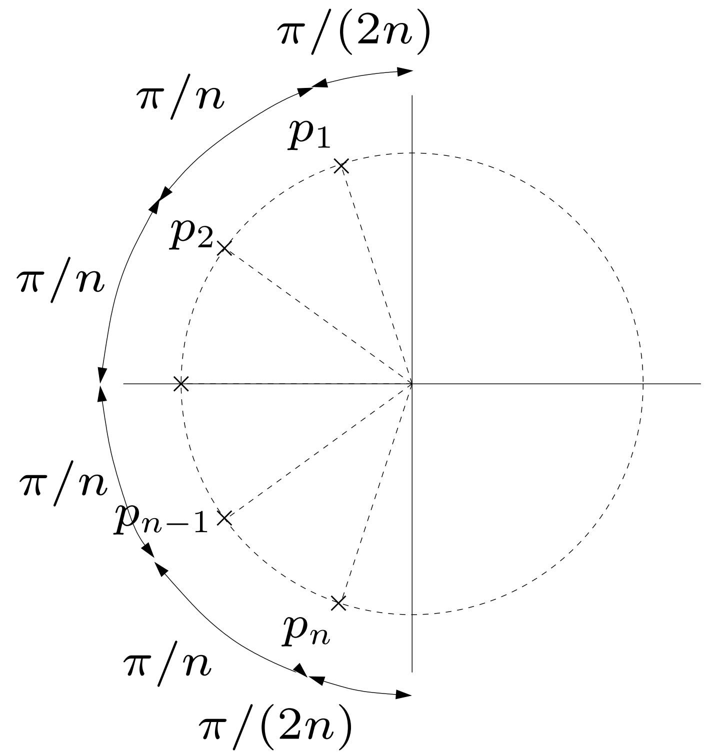
$$|H(j\omega)| = 1 \text{ for } 0 \leq \omega \leq 1, \quad |H(j\omega)| = 0 \text{ for } \omega > 1$$

by a *rational* transfer function (which can be synthesized using R, L, C)

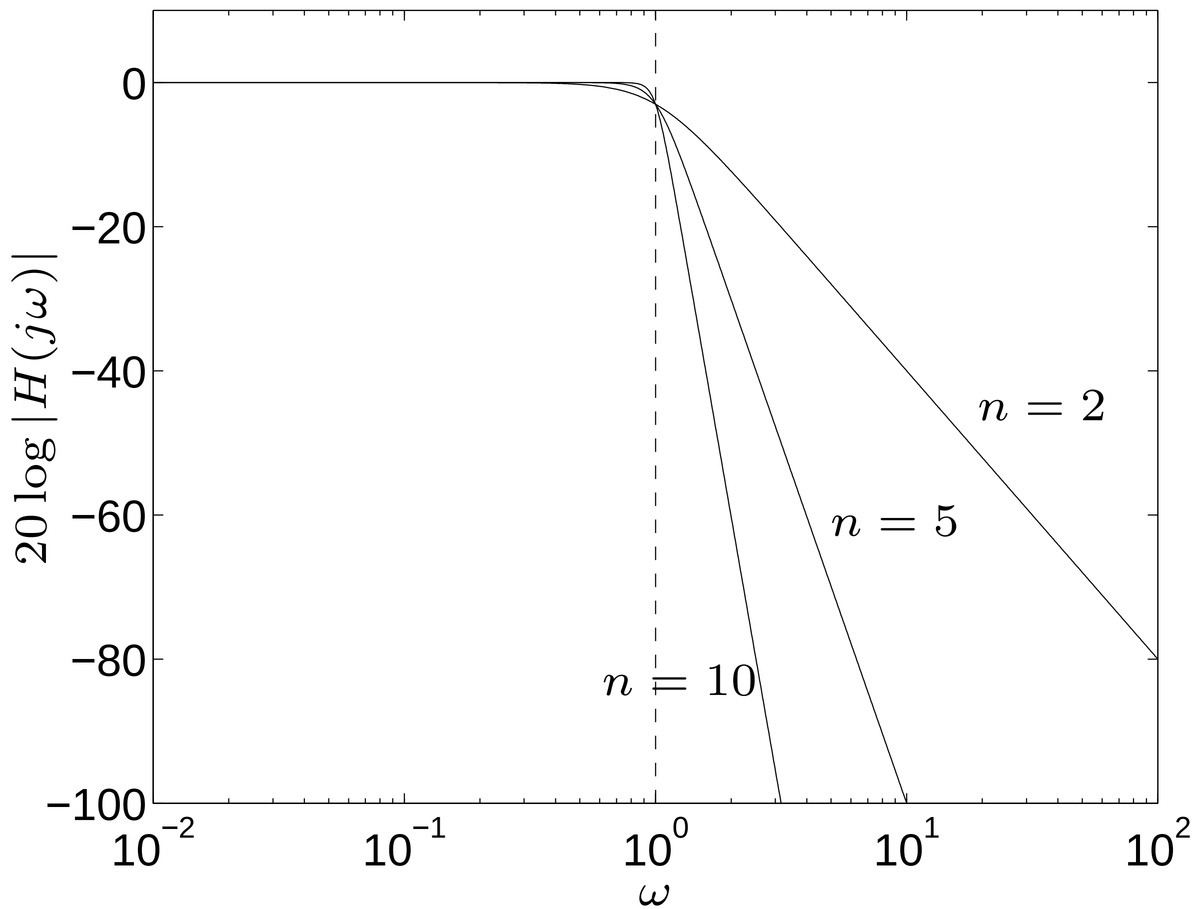
example nth-order Butterworth filter

$$H(s) = \frac{1}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

n stable poles, equally spaced on the unit circle



magnitude plot for $n = 2$, $n = 5$, $n = 10$



Sketching approximate Bode plots

sum property allows us to find Bode plots of terms in TF, then add

simple terms:

- constant
- factor of s^k (pole or zero at $s = 0$)
- real pole, real zero
- complex pole or zero pair

from these we can construct Bode plot of any rational transfer function

Poles and zeros at $s = 0$

the term s^k has simple Bode plot:

- phase is constant, $\angle = 90k^\circ$
- magnitude has constant slope $20k$dB/decade
- magnitude plot intersects 0dB axis at $\omega = 1$

examples:

- integrator $(k = -1)$: $\angle = -90^\circ$, slope is -20dB/decade
- differentiator $(k = +1)$: $\angle = 90^\circ$, slope is $+20$dB/decade

Real poles and zeros

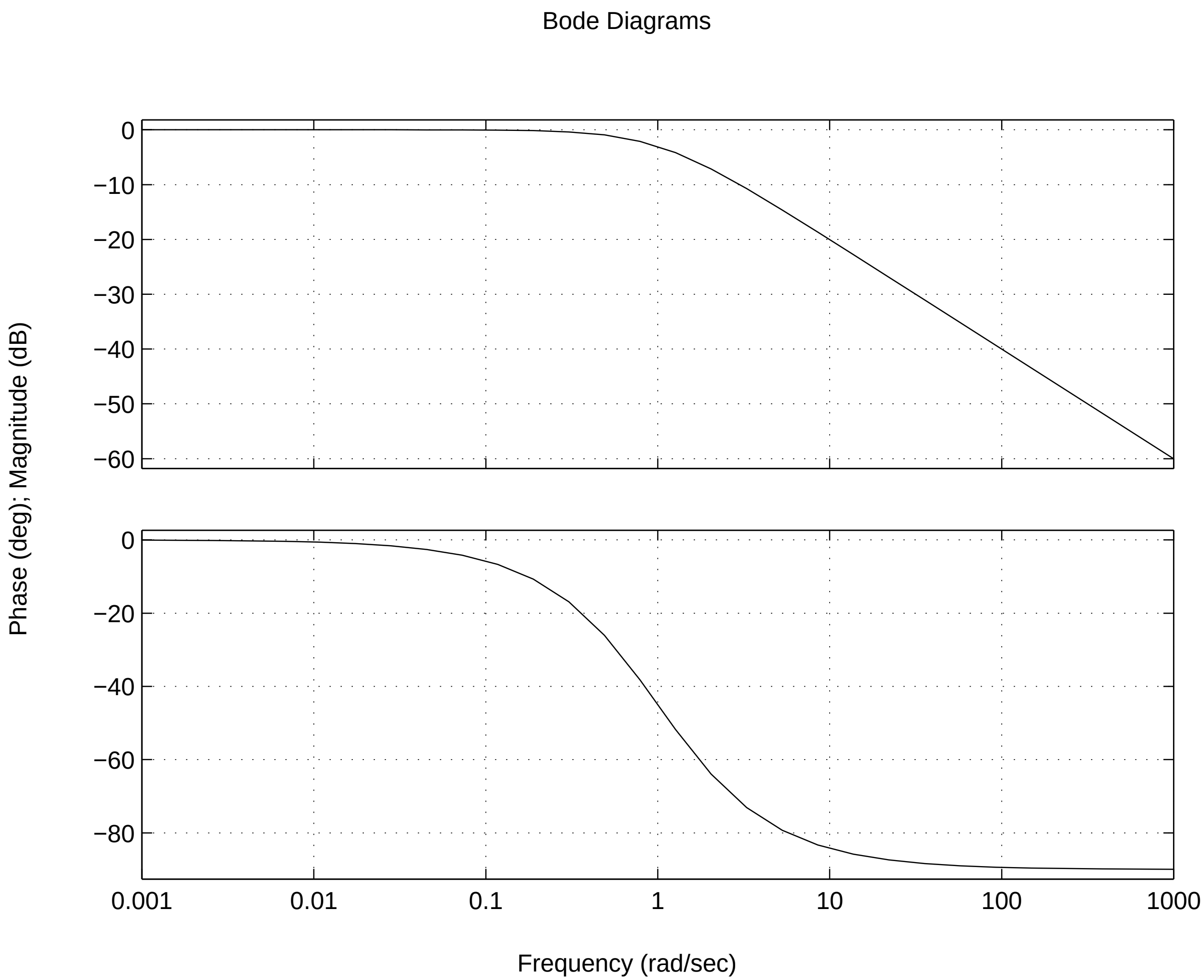
$H(s) = 1/(s-p)$ ($p < 0$ is stable pole; $p > 0$ is unstable pole)

magnitude: $|H(j\omega)| = 1/\sqrt{\omega^2 + p^2}$

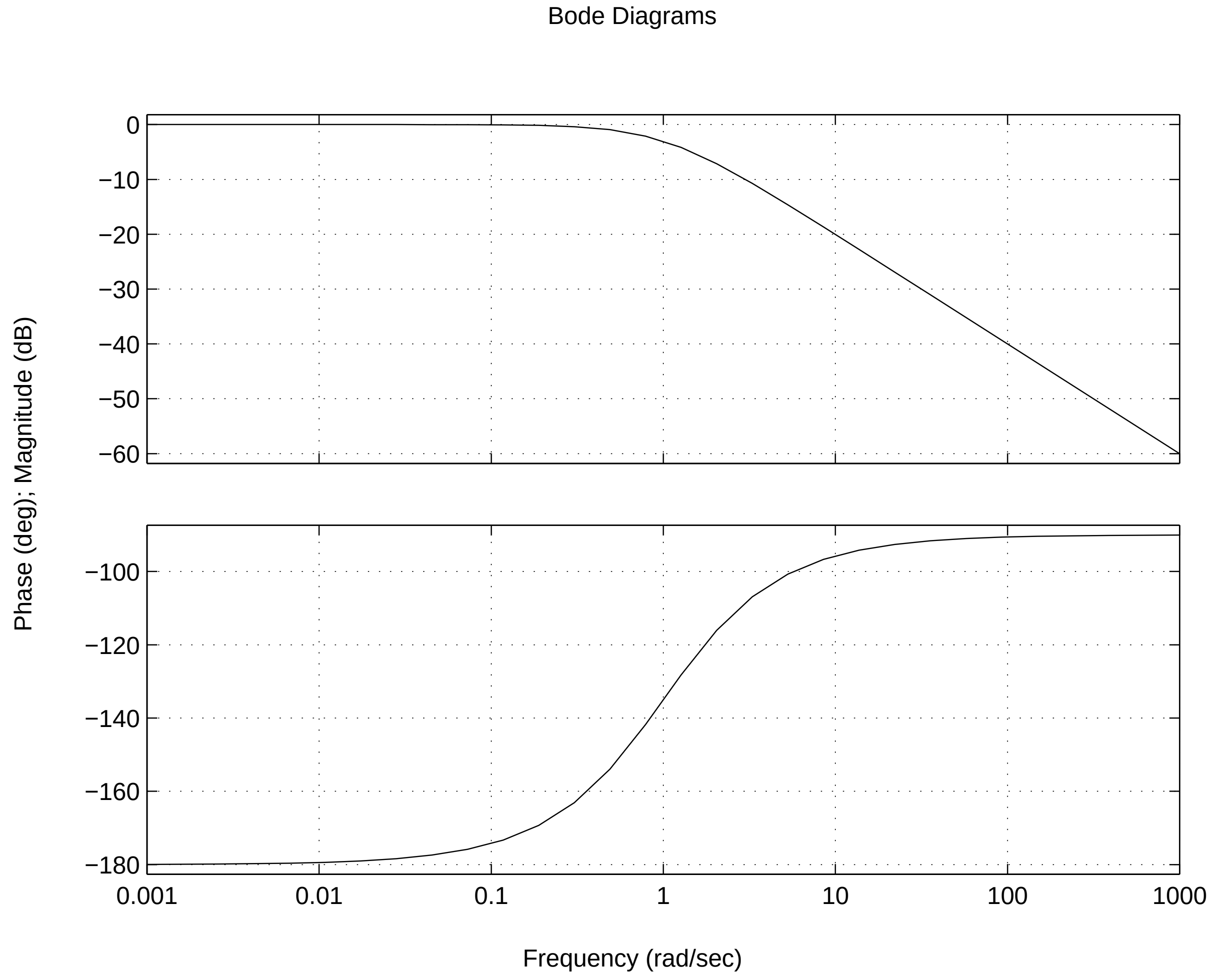
for $p < 0$, $\angle H(j\omega) = -\arctan(\omega/|p|)$

for $p > 0$, $\angle H(j\omega) = \pm 180^\circ + \arctan(\omega/p)$

Bode plot for $H(s) = 1/(s+1)$ (stable pole):



Bode plot for $H(s) = 1/(s - 1)$ (unstable pole):

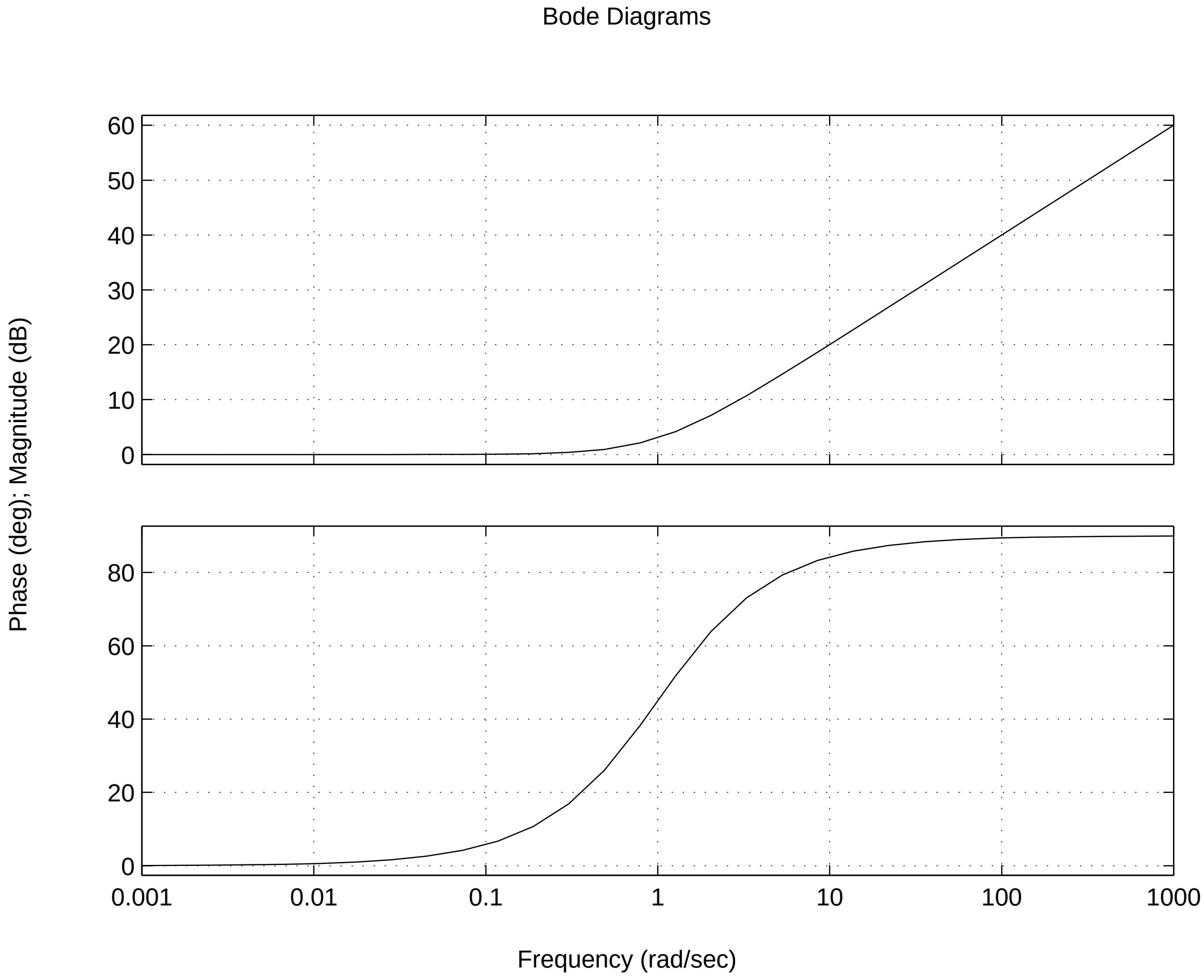


magnitude same as stable pole; phase starts at $-180°$, increases *straight-line* approximation (for $p < 0$):

- for $\omega < |p|$, $|H(j\omega)| \approx 1/|p|$
- for $\omega > |p|$, $|H(j\omega)|$ decreases ('falls off') 20dB per decade
- for $\omega < 0.1|p|$, $\angle H(j\omega) \approx 0$
- for $\omega > 10|p|$, $\angle H(j\omega) \approx -90^\circ$
- in between, phase is approximately linear (on log-log plot)

Bode plots for real zeros same as poles but upside down

example: $H(s) = s + 1$



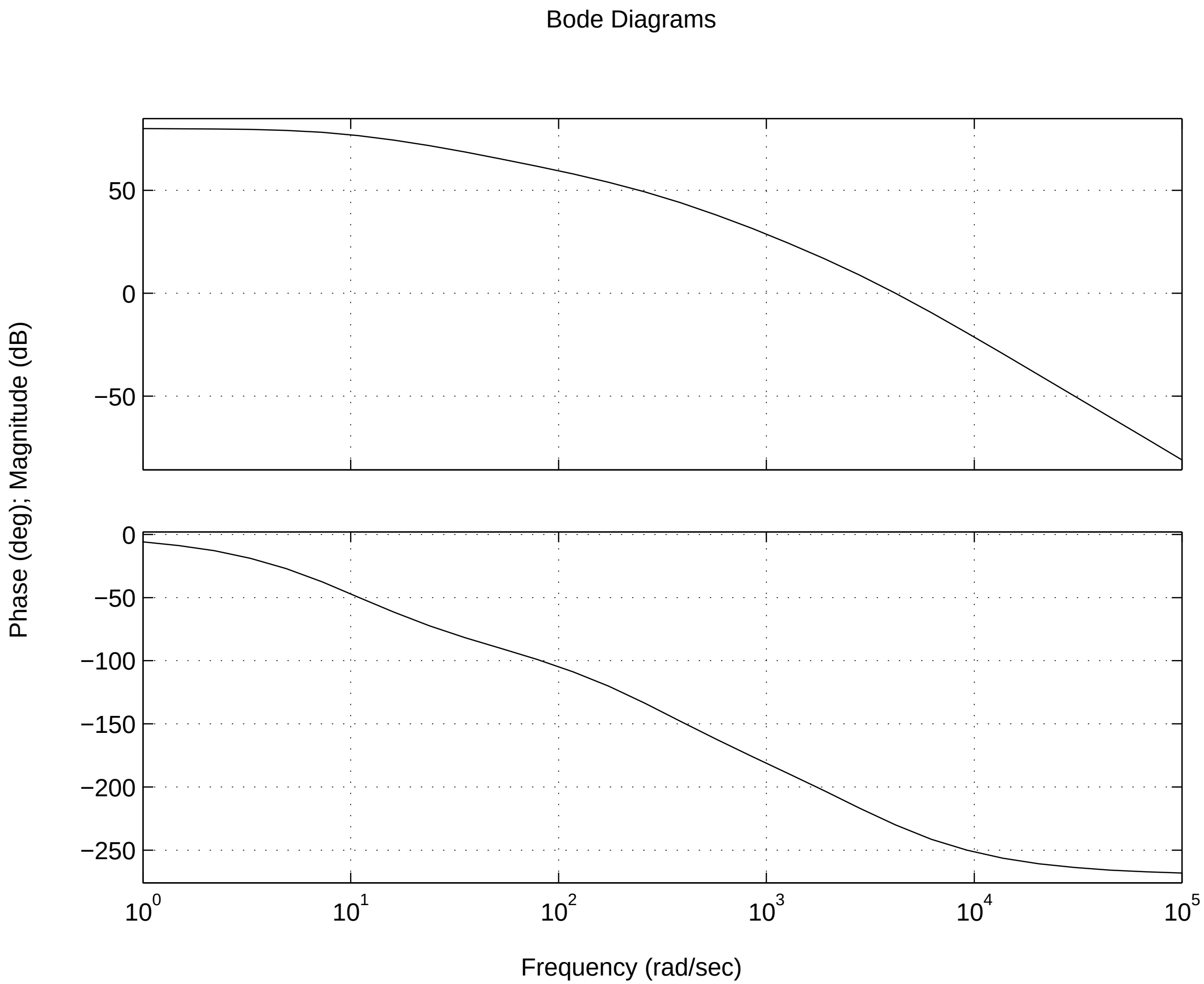
example:

$$H(s) = \frac{10^4}{(1+s/10)(1+s/300)(1+s/3000)}$$

(typical op-amp transfer function)

DC gain 80dB; poles at -10, -300, -3000

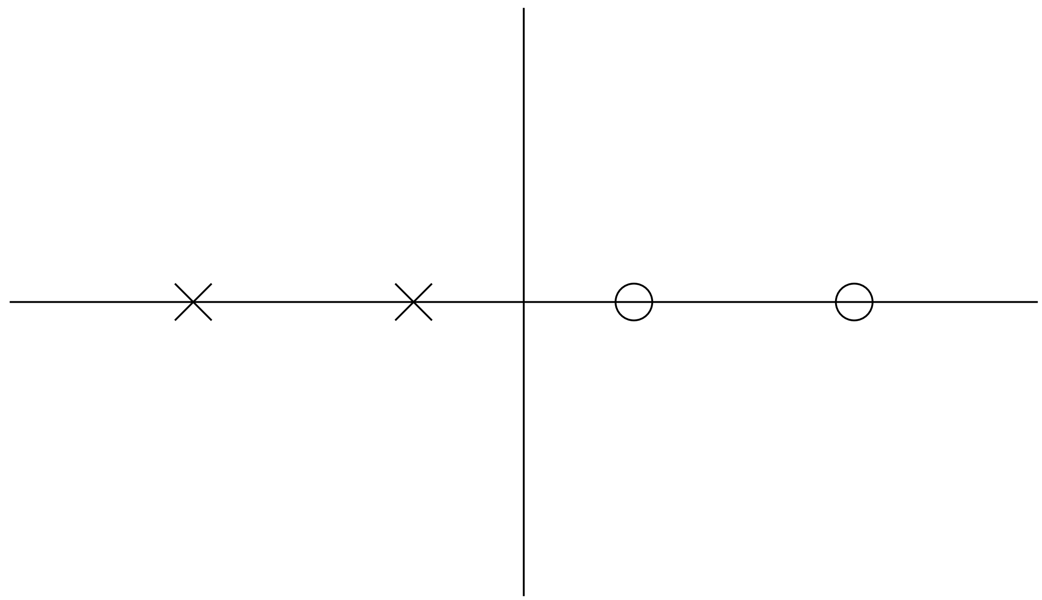
Bode plot:



Bode Diagrams
Phase (deg); Magnitude (dB)
50
0
−50
0
−50
−100
−150
−200
−250
10^0
10^1
10^2
10^3
10^4
10^5
Frequency (rad/sec)

example:

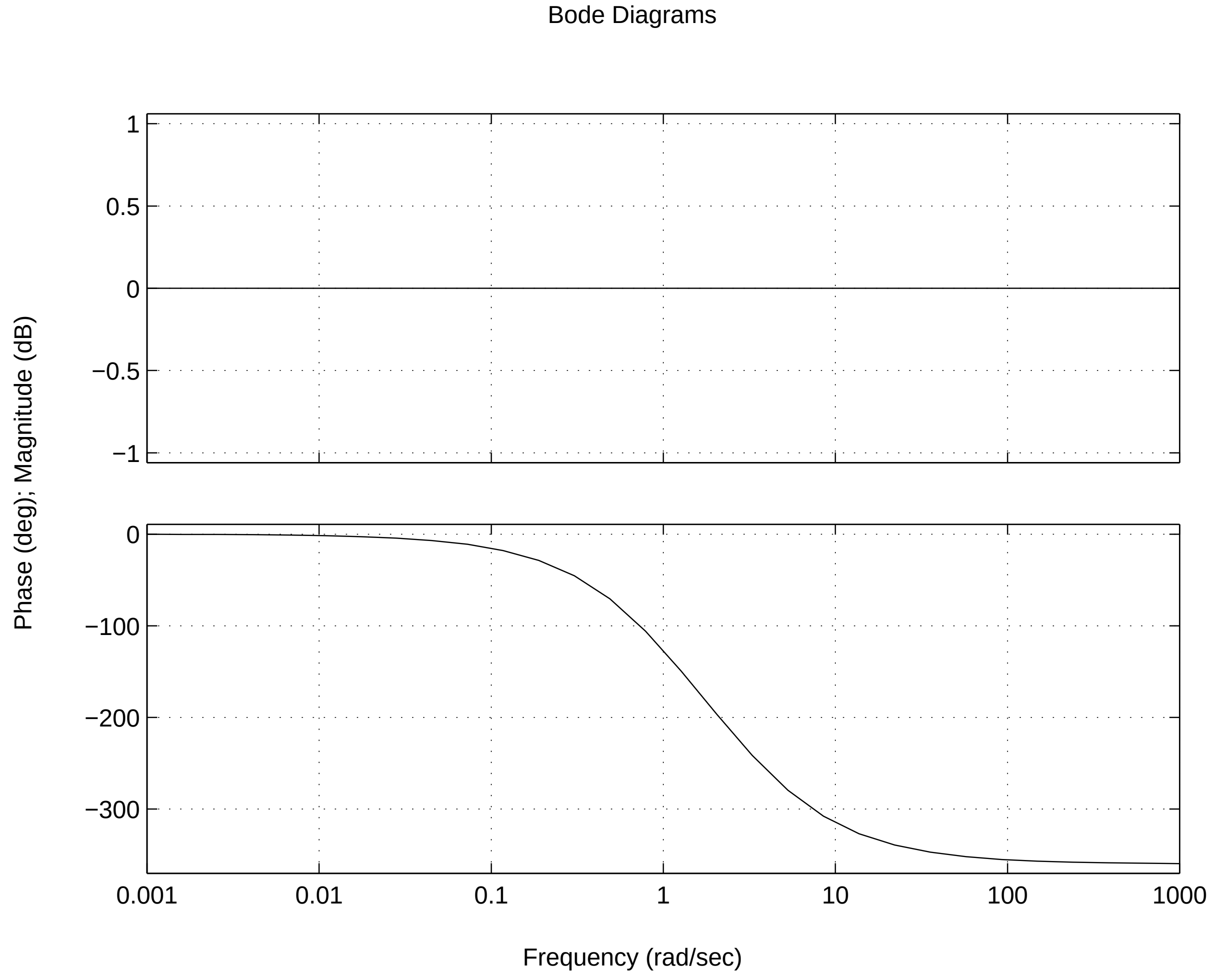
$$H(s)=\frac{(s-1)(s-3)}{(s+1)(s+3)}=\frac{s^2-4s+3}{s^2+4s+3}$$



$|H(j\omega)|=1$ (obvious from graphical interpretation)

(called *all-pass filter* or *phase filter* since gain magnitude is one for all frequencies)

Bode plot:



Bode Diagrams
Phase (deg); Magnitude (dB)
1
0.5
0
−0.5
−1
0
−100
−200
−300
0.001
0.01
0.1
1
10
100
1000
Frequency (rad/sec)

High frequency slope

$$H(s) = \frac{b_0 + \cdots b_m s^m}{a_0 + \cdots + a_n s^n}$$

$b_m, a_n \neq 0$

for ω large, $H(j\omega) \approx (b_m/a_n)(j\omega)^{m-n}$, *i.e.*,

$$20 \log_{10} |H(j\omega)| \approx 20 \log_{10} |b_m/a_n| - (n-m) 20 \log_{10} \omega$$

- high frequency magnitude slope is approximately $-20(n-m)$dB/decade
- high frequency phase is approximately $\angle(b_m/a_n) - 90(n-m)^\circ$