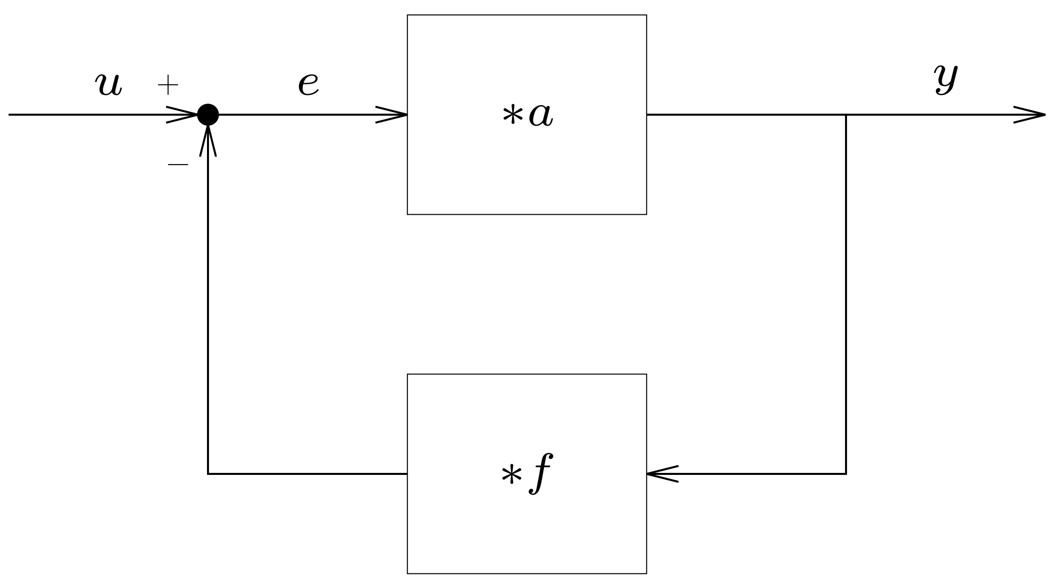
Lecture 13
Dynamic analysis of feedback

- Closed-loop, sensitivity, and loop transfer functions
- Stability of feedback systems

Some assumptions

we now assume:

- signals u, e, y are *dynamic*, $i.e.$, change with time
- open-loop and feedback systems are convolution operators, with impulse responses a and f, respectively



feedback equations are now:

$$y(t) = \int_0^t a(\tau)e(t-\tau)\, d\tau, \qquad e(t) = u(t) - \int_0^t f(\tau)y(t-\tau)\, d\tau$$

- these are *complicated* (integral equations)
- it's not so obvious what to do — current input $u(t)$ affects future output $y(\bar{t})$, $\bar{t} \geq t$

Feedback system: frequency domain

take Laplace transform of all signals:

$$Y(s) = A(s)E(s), \qquad E(s) = U(s) - F(s)Y(s)$$

eliminate $E(s)$ (just algebra!) to get

$$Y(s) = G(s)U(s), \qquad G(s) = \frac{A(s)}{1 + A(s)F(s)}$$

G is called the *closed-loop transfer function*

. . . exactly the same formula as in static case, but now A, F, G are transfer functions we define

- loop transfer function $L = AF$
- sensitivity transfer function $S = 1/(1 + AF)$

same formulas as static case!

for example, for small δA, we have

$$\frac{\delta G}{G} \approx S \frac{\delta A}{A}$$

(but these are transfer functions here)

what's new: L, S, G

- depend on frequency s
- are complex-valued
- can be stable or unstable

thus:

- "large" and "small" mean complex magnitude
- L (or G or S) can be large for some frequencies, small for others
- step response of G shows time response of the closed-loop system

Example

feedback system with

$$A(s) = \frac{10^5}{1 + s/100}, \qquad F = 0.01$$

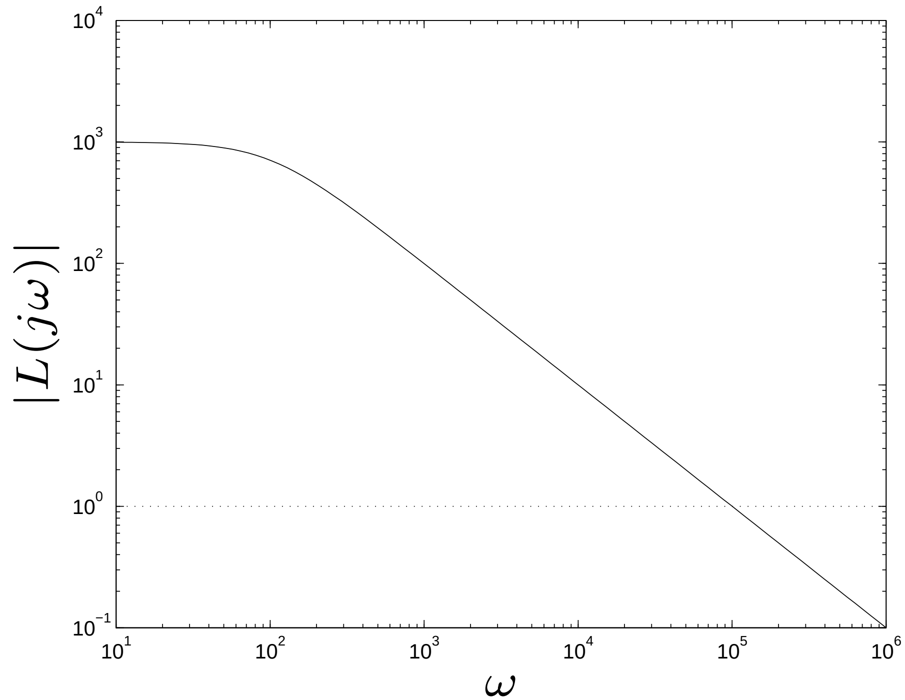
- open-loop gain is large at DC (10^5)
- open-loop bandwidth is around 100 rad/sec
- open-loop settling time is around 20msec

closed-loop transfer function is

$$G(s) = \frac{\frac{10^5}{1+s/100}}{1 + 0.01\frac{10^5}{1+s/100}} = \frac{99.9}{1 + s/(1.001 \cdot 10^5)}$$

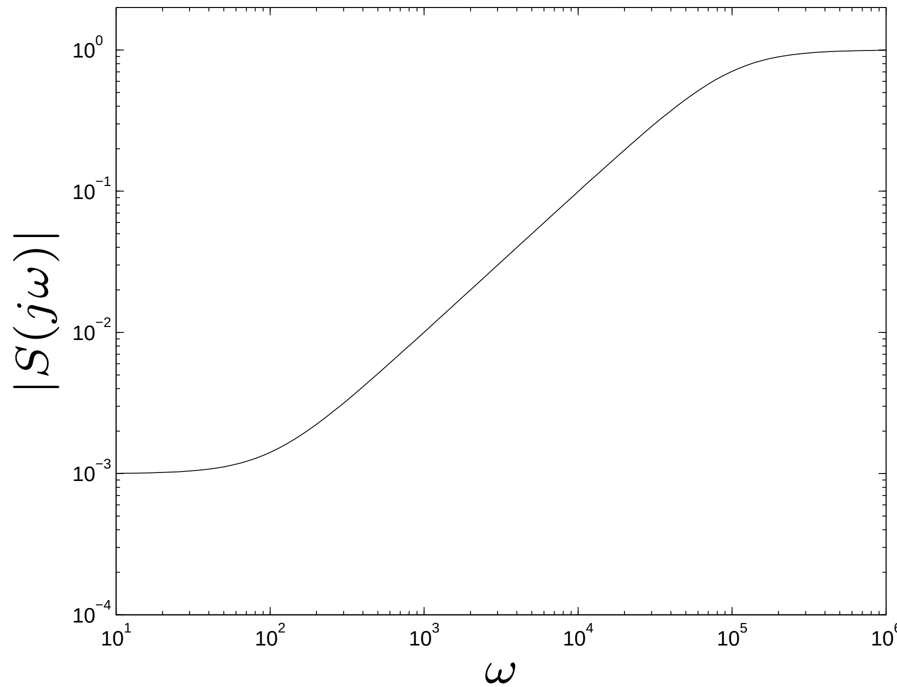
- G is stable
- closed-loop DC gain is very nearly $1/F$
- closed-loop bandwidth around 10^5 rad/sec
- closed-loop settling time is around 20μsec

. . . closed-loop system has lower gain, higher bandwidth, *i.e.*, is faster loop transfer function is $L(s) = \dfrac{10^3}{1 + s/100}$, so $|L(j\omega)| = \dfrac{10^3}{\sqrt{1 + (\omega/100)^2}}$



- loop gain larger than one for $\omega < 10^5$ or so
  $\Rightarrow$ get benefits of feedback for $\omega < 10^5$
- loop gain less than one for $\omega > 10^5$ or so
  $\Rightarrow$ don't get benefits of feedback for $\omega > 10^5$

sensitivity transfer function is $S(s) = \dfrac{1 + s/100}{1001 + s/100}$



- $|S(j\omega)| \ll 1$ for $\omega < 10^4$ (say)
- $|S| \approx 1$ for $\omega > 10^5$ or so

Thus, $e.g.$, for small changes in $A(0)$, $A(j10^5)$

$$\left|\frac{\delta G(0)}{G(0)}\right| \approx 10^{-3}\left|\frac{\delta A(0)}{A(0)}\right|, \quad \left|\frac{\delta G(j10^5)}{G(j10^5)}\right| \approx \left|\frac{\delta A(j10^5)}{A(j10^5)}\right|$$

Example (with change of sign)

now consider system with $A(s) = -\frac{10^5}{1+s/100}$, $F = 0.01$
(note minus sign!)

closed-loop transfer function is

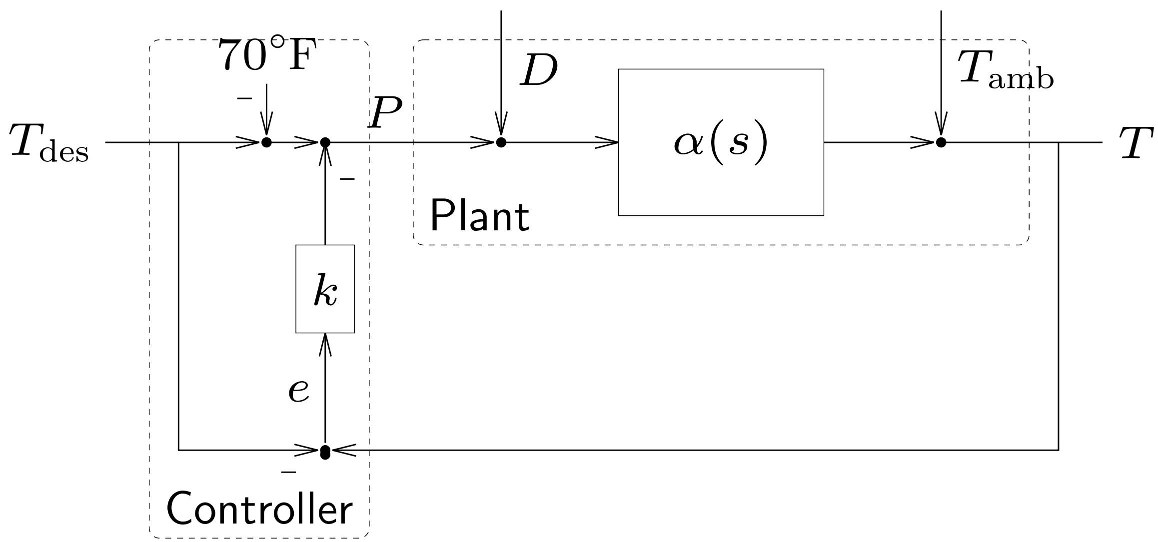
$$G(s) = \frac{100.1}{1 - s/(0.999 \cdot 10^5)}$$

looks like G found above, but is **unstable**

- in static analysis, large loop gain $\Rightarrow$ sign of feedback doesn't much matter
- dynamic analysis reveals the big difference a change of sign can make

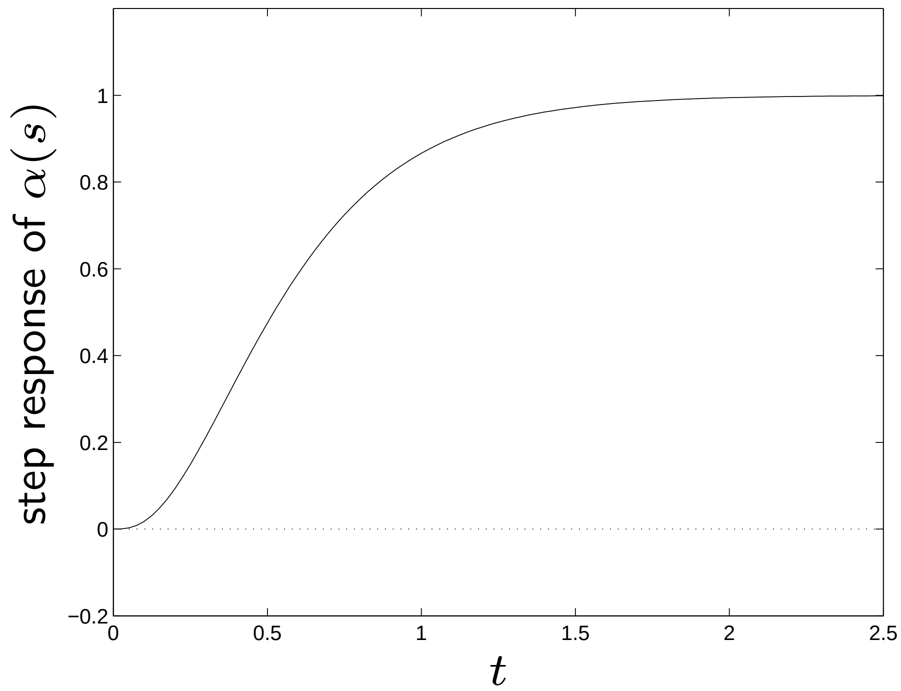
Heater example: dynamic analysis

proportional controller of lecture 12,



with *dynamic model* of plate:

$$\alpha(s) = \frac{1}{(1+0.1s)(1+0.2s)(1+0.3s)}$$



(quite realistic; takes about 1 sec to heat up)

Let's assume

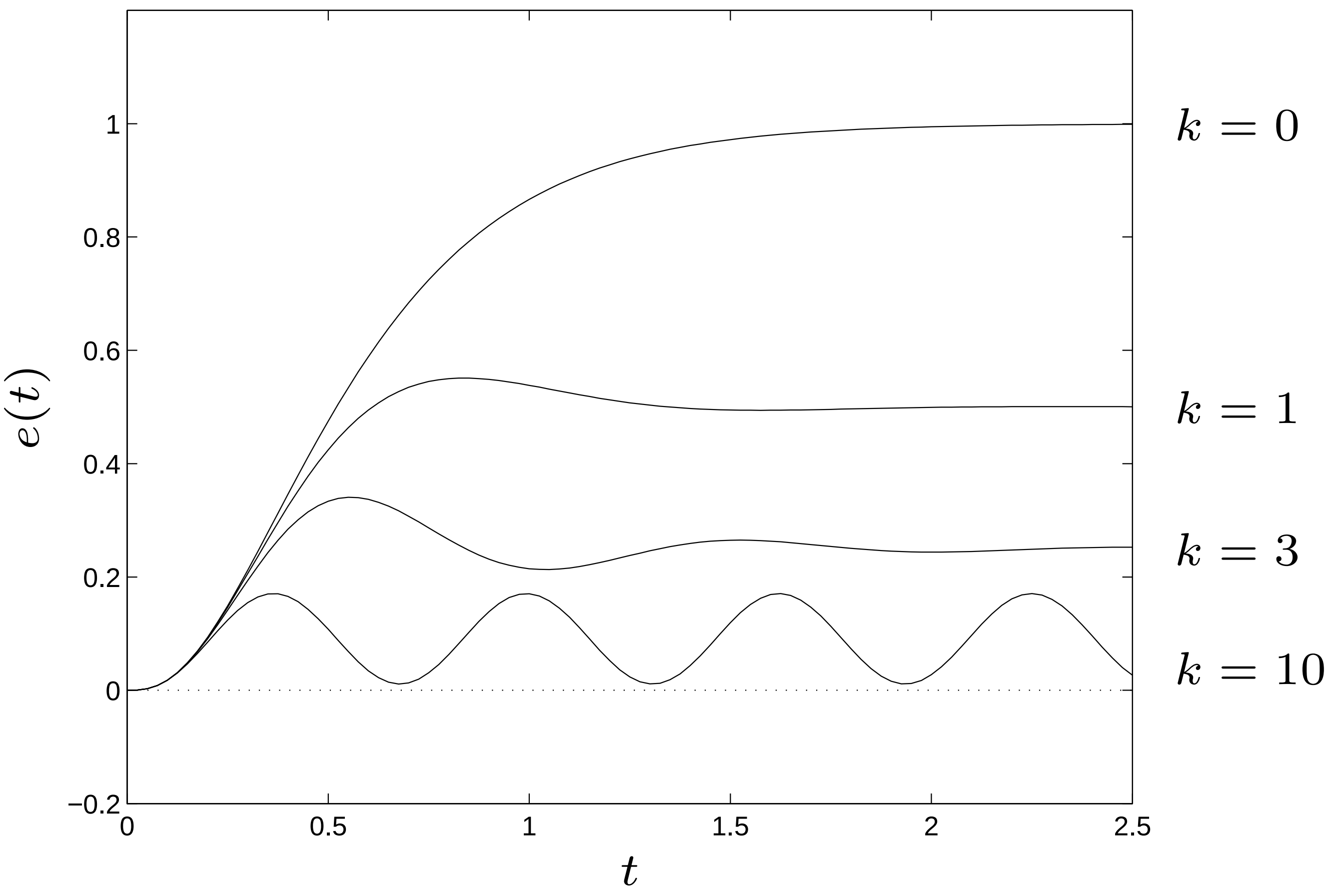
- $T_{\mathrm{amb}} = 70°\mathrm{F}$
- $T_{\mathrm{des}} = 150°\mathrm{F}$ (actually doesn't matter)
- D is a unit step, *i.e.*, for $t \geq 0$ a disturbance power of $1\mathrm{W}$ is applied
- for $t < 0$ system is in static steady-state (with $T = T_{\mathrm{des}}$)

$\Rightarrow$ have an LTI system from D to temperature error e;

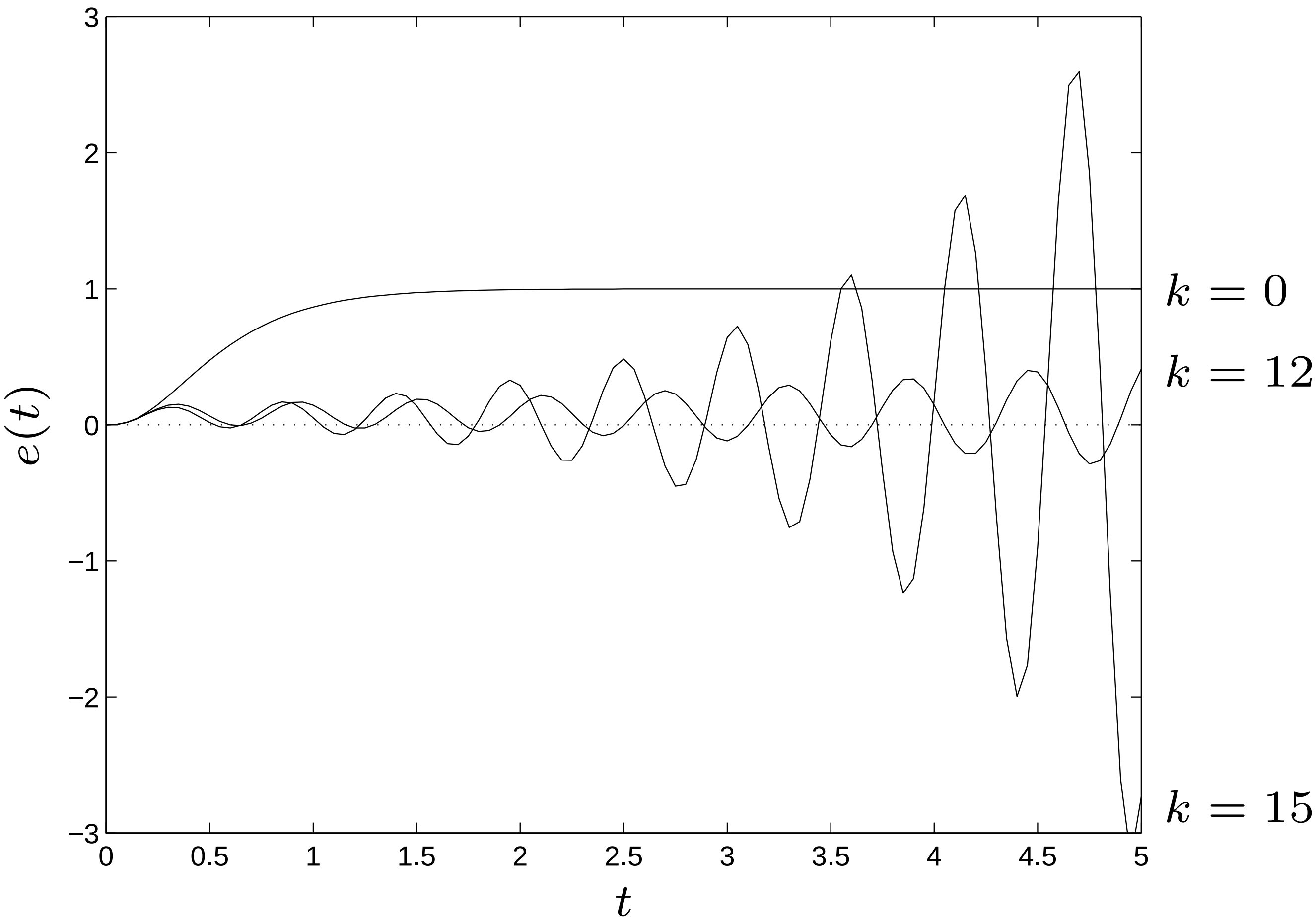
transfer function is

$$\frac{\alpha(s)}{1 + k\alpha(s)}$$

step response gives temperature error resulting from unit step disturbance power



$e(t)$
t
$k = 0$
$k = 1$
$k = 3$
$k = 10$
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.5
1
1.5
2
2.5


3
2
1
0
−1
−2
−3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
e(t)
t
k = 0
k = 12
k = 15

- closed-loop system can exhibit oscillatory response
- for $k < 10$ (approximately) this transfer function is stable; for $k > 10$ (approximately) it is unstable
- when stable, step response settles to DC gain, $1/(1 + k)$
- *stability* requirement limits how large proportional gain (hence loop gain) can be

these are general phenomena

Design: choice of k

involves tradeoff of static sensitivity, $1/(1+k)$, versus dynamic response

- $k < 1$ (or so) $\Rightarrow$ closed-loop system not much better than open-loop
- $k > 5$ (or so) $\Rightarrow$ undesirable oscillatory response
- $k > 10$ (or so) $\Rightarrow$ very undesirable instability

. . . here, maybe $k = 2$ or 3 is about right

Let's do some analysis . . .

transfer function from D to e is

$$\frac{\alpha(s)}{1+k\alpha(s)} = \frac{1}{(1+0.1s)(1+0.2s)(1+0.3s)+k}$$

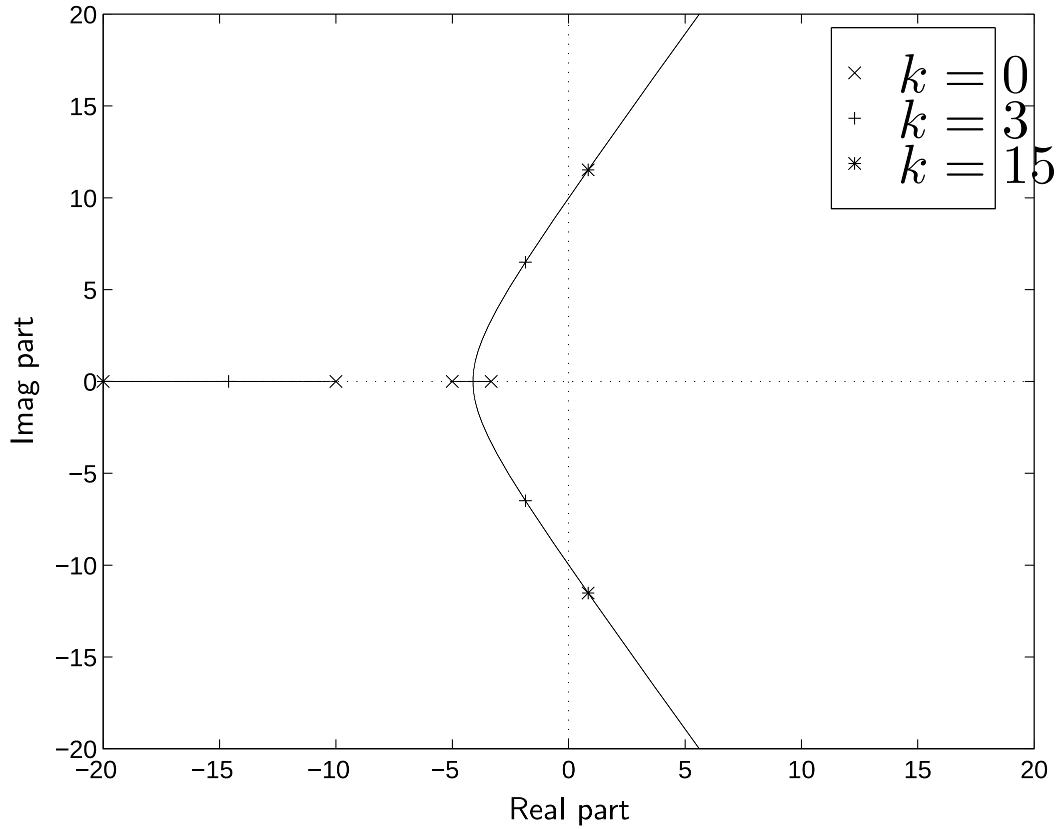
its poles are the roots of the polynomial

$$(1 + 0.1s)(1 + 0.2s)(1 + 0.3s) + k,$$

which of course depend on k

k	poles	
0	-10.0,	-5.00, -3.33
1	-12.5,	$-2.94 \pm 4.26j$
3	-14.6,	$-1.86 \pm 6.49j$
10	-18.3,	$\pm 10.0j$
12	-19.1,	$+0.36 \pm 10.7j$
15	-20.0,	$+0.83 \pm 11.5j$

poles are often plotted on complex plane:



called *root locus* plot of

$$(1+0.1s)(1+0.2s)(1+0.3s)+k$$

Checking stability

when is $H(s) = b(s)/a(s)$ stable?

$i.e.$, when do all roots of the polynomial a have negative real parts (such polynomials are called *Hurwitz*)

if a is already factored, as in

$$a(s) = \alpha(s - p_1)(s - p_2) \cdots (s - p_n),$$

we just check $\Re(p_i) < 0$ for $i = 1, \ldots, n$

what if we are given the coefficients of a:

$$a(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$$

if the a_i's are specific numbers, we can easily factor a numerically (using a computer), then check

but what if the coefficients involve parameters, as in

$$a(s) = (1 + 0.1s)(1 + 0.2s)(1 + 0.3s) + k$$

can we get the roots p_i in terms of the coefficients a_i? . . . an old problem

- there are analytical formulas for the roots of a polynomial, for degrees 1, 2, 3, and 4 (they are complicated for third and fourth degree)
- there are *no analytical formulas* for the roots of a polynomial of degree ≥ 5 (a famous result of Galois)

still, it turns out that we can express the Hurwitz condition as a set of algebraic inequalities involving the coefficients, using Routh's method (1870 or so)

- very useful 50 years ago, even for polynomials with specific numeric coefficients
- only important nowadays for polynomials with parameters

we assume that $a_n = 1$ (if not, divide $a(s)$ by a_n; doesn't affect roots) so we have $a(s) = a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n$

Fact: a is Hurwitz $\Rightarrow a_0 > 0, \ldots, a_{n-1} > 0$

to see this, write a in real factored form:

$$\begin{aligned} a(s) &= a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n \\ &= \prod_{i=1}^{q}(s - p_i) \cdot \prod_{i=1}^{r}(s^2 - 2\sigma_i s + \sigma_i^2 + \omega_i^2) \end{aligned}$$

p_i are the real roots, $\sigma_i \pm j\omega_i$ are the complex roots of a

Hurwitz means $p_i < 0$ and $\sigma_i < 0$, so each term is a polynomial with positive coefficents

a is a product of polynomials with all positive coefficients, hence has all positive coefficients

the converse is *not* true: *e.g.*, $a(s) = s^3 + s^2 + s + 2$ has roots -1.35, $+0.177 \pm 1.2j$, so it's not Hurwitz

Hurwitz conditions

(obtained from Routh's method or formulas for roots)

- *Degree 1:* $a_0 + s$ is Hurwitz $\Leftrightarrow\; a_0 > 0$
- *Degree 2:* $a_0 + a_1 s + s^2$ is Hurwitz

$$\Leftrightarrow\; a_0 > 0,\; a_1 > 0$$

- *Degree 3:* $a_0 + a_1 s + a_2 s^2 + s^3$ is Hurwitz

$$\begin{aligned} \Leftrightarrow\quad & a_0 > 0,\; a_1 > 0,\; a_2 > 0, \\ & a_2 a_1 > a_0 \end{aligned}$$

- *Degree 4:* $a_0 + a_1 s + a_2 s^2 + a_3 s^3 + s^4$ is Hurwitz

$$\Leftrightarrow\quad a_0 > 0,\; a_1 > 0,\; a_2 > 0,\; a_3 > 0,$$

$$a_3 a_2 > a_1,$$
$$a_1 a_2 a_3 - a_3^2 a_0 > a_1^2$$

for degree ≥ 5, conditions get much more complex

- you can find them via Routh's method, if you need to (you probably won't)
- they consist of inequalities involving sums & products of the coefficients

Application: for what values of proportional gain k is our example, the plate heating system, stable?

I.e., for what values of k is

$$\begin{aligned} a(s) &= (1+0.1s)(1+0.2s)(1+0.3s)+k \\ &= 0.006(167(k+1)+100s+18.3s^2+s^3) \end{aligned}$$

Hurwitz?

Hurwitz conditions are:

$$167(k+1)>0,\ 100>0,\ 18.3>0,$$

$$100\cdot 18.3 > 167(k+1),$$

which simplify to: $-1<k<10$

(we suspected this from our numerical studies)

Summary

for LTI feedback systems,

- formulas same as static case, but now A, F, L, S are transfer functions
- hence are complex, depend on frequency s, and can be stable or unstable
- stability requirement often limits the amount of feedback that can be used