Application-specific architectures for large finite element applications

by Valerie Taylor, 1991

Philip Levis
Stanford CS349G
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(from last time)

Fundamental Challenge

- A lot of the field of computer graphics struggles with the challenges of quantizing the continuous
  - Line are lines, but we have to turn them into pixels
  - Quantizing a curved surface into triangles
  - Simulating fluids, materials, sound
- Related to, but different, from high-performance computing (Taylor's dissertation)
  - High performance computing is about accurately predicting the physical world (chemical reactions, structural stress/strain, heat transfer, explosions, weather, etc.)
  - Graphics is about making something that looks convincing

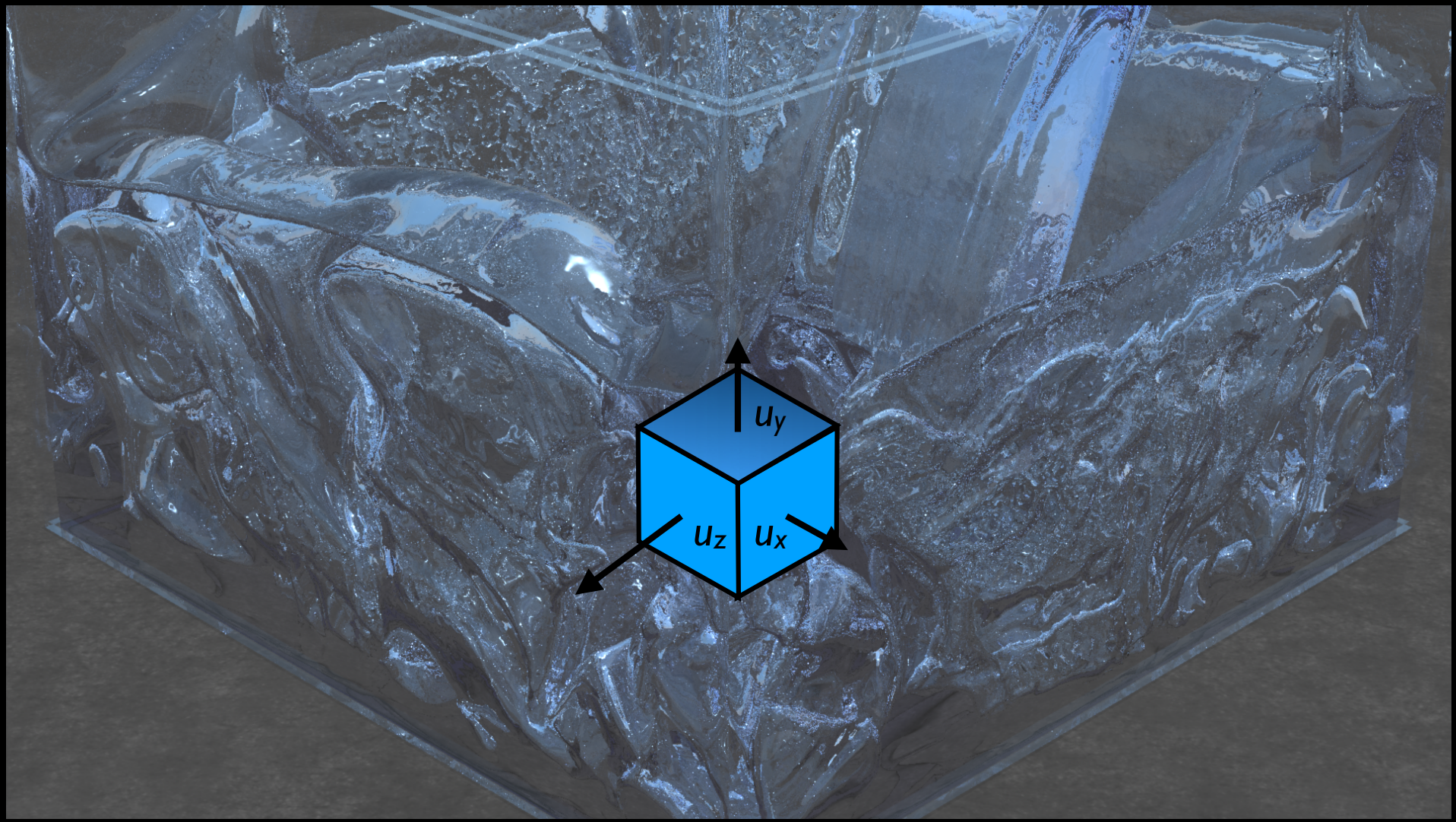
Simulation

- Want to numerically (rather than analytically) simulate what will happen to a physical system
- Discretize the system to a set of data points
  - Example: simulate water as grid cells, each cell stores whether it has water, the water's velocity, the water's pressure
- Use equations to specify how cells affect one another, take time steps
  - Example: Navier-Stokes equations for fluid flow

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \frac{\nabla p}{\rho} + v\nabla^2 u$$

movement of velocity field — advection — pressure — diffusion

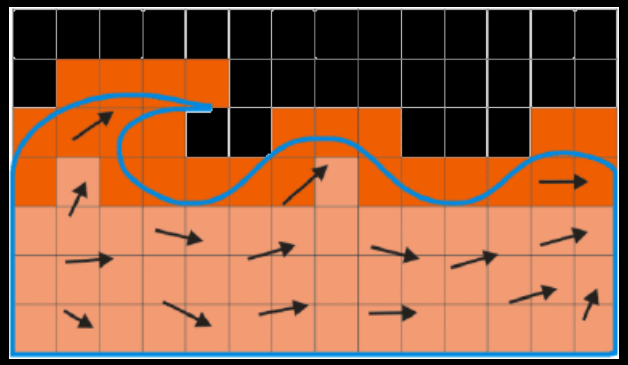
Discretizing the Continuous



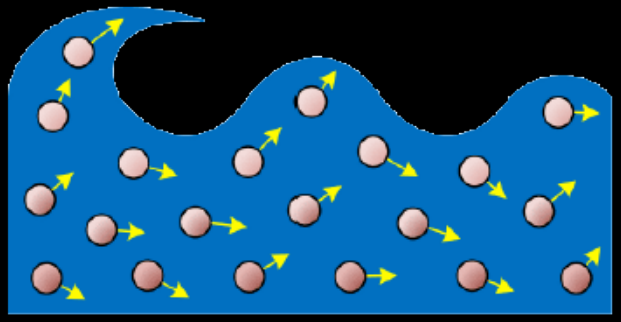
Discretization

- Equations are continuous, use partial derivatives: $\frac{\partial u}{\partial t}$
- Need to quantize/sample the space, derivatives are differences between sample points

Three Approaches



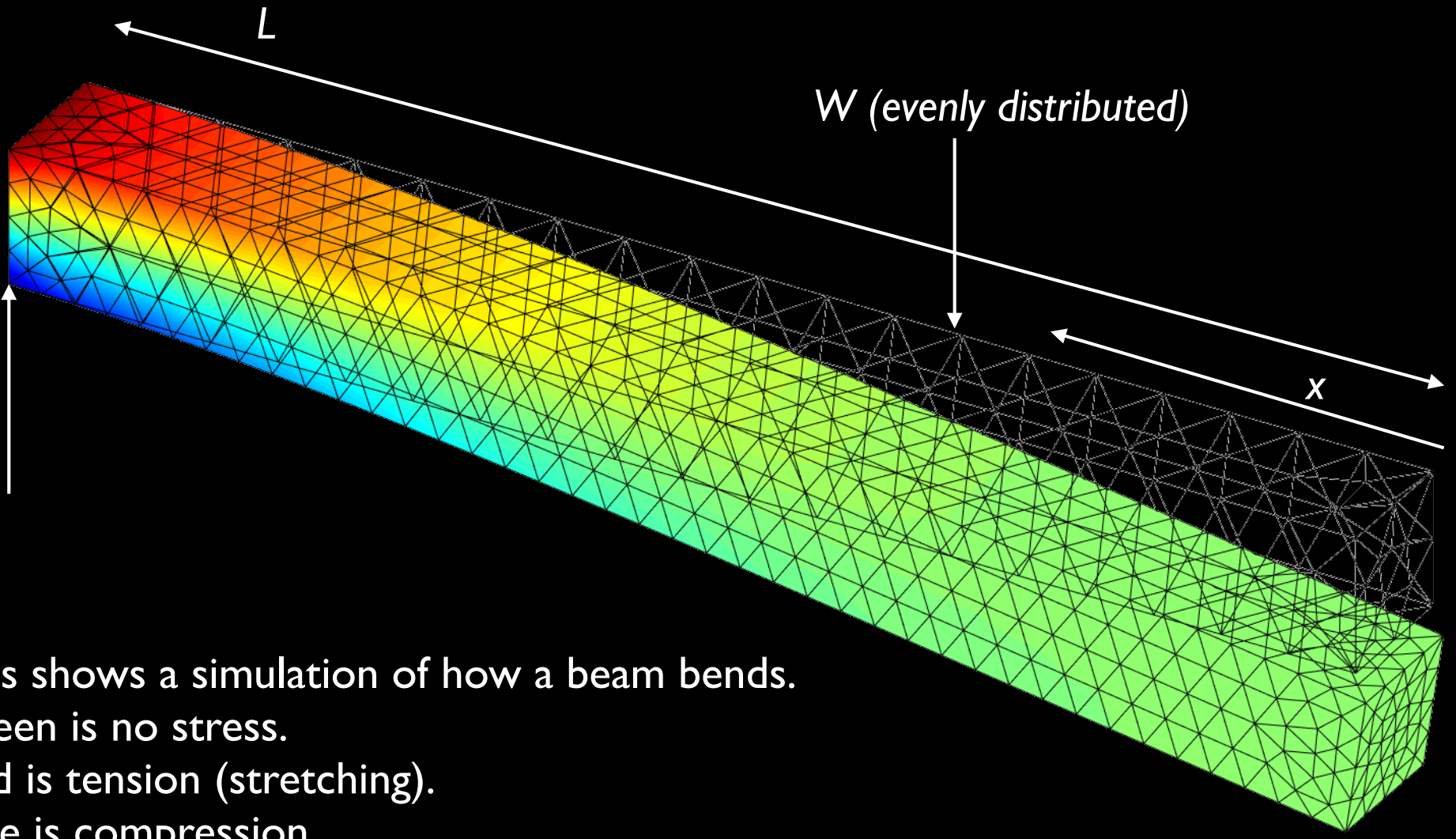
Eulerian



Lagrangian

Mesh

Beam Bending



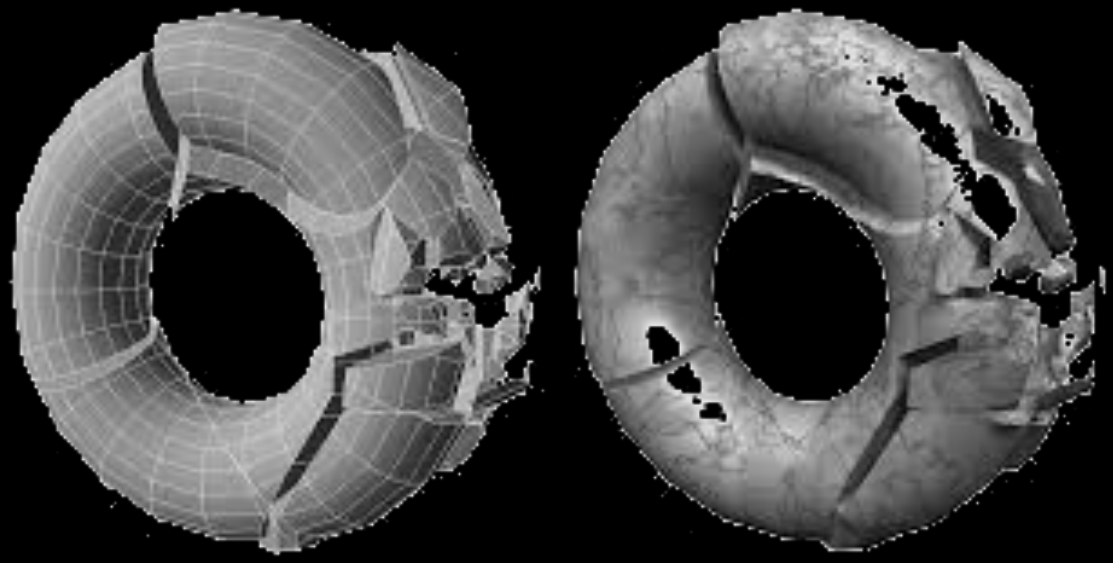
This shows a simulation of how a beam bends.
Green is no stress.
Red is tension (stretching).
Blue is compression.

$$\Delta x = \frac{w}{48EI}(x^4 - 4L^3x - 3x^4)$$

E is elasticity modulus
I is 2nd moment of area

Need for Simulation

- Equations like the cantilever beam under uniform load are closed form solutions because the system is so simple.
- What if there are highly varying forces, time-varying forces, and complex objects?

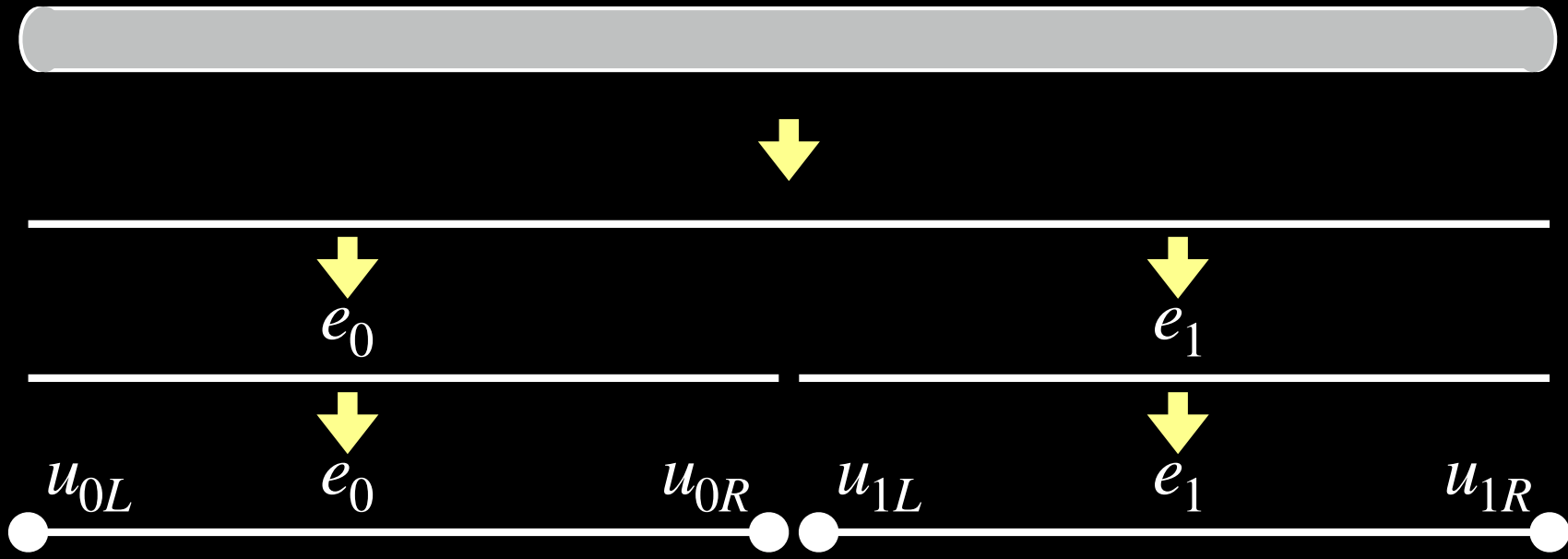


Finite Element Method

- Represent a complex object to a *finite* set of elements, defined by a mesh
- Define the state of each element by equations based on its state, neighboring elements, and external forces.
- The combination of the mesh relationships and equations create a sparse system of linear equations
- Using the finite element method involves solving these equations to simulate what happens
- Valerie Taylor's dissertation is about a hardware accelerator to solve them faster

Finite Element Method

- Let's consider a trivial 1D case for simplicity
- Want to estimate how this bar bends



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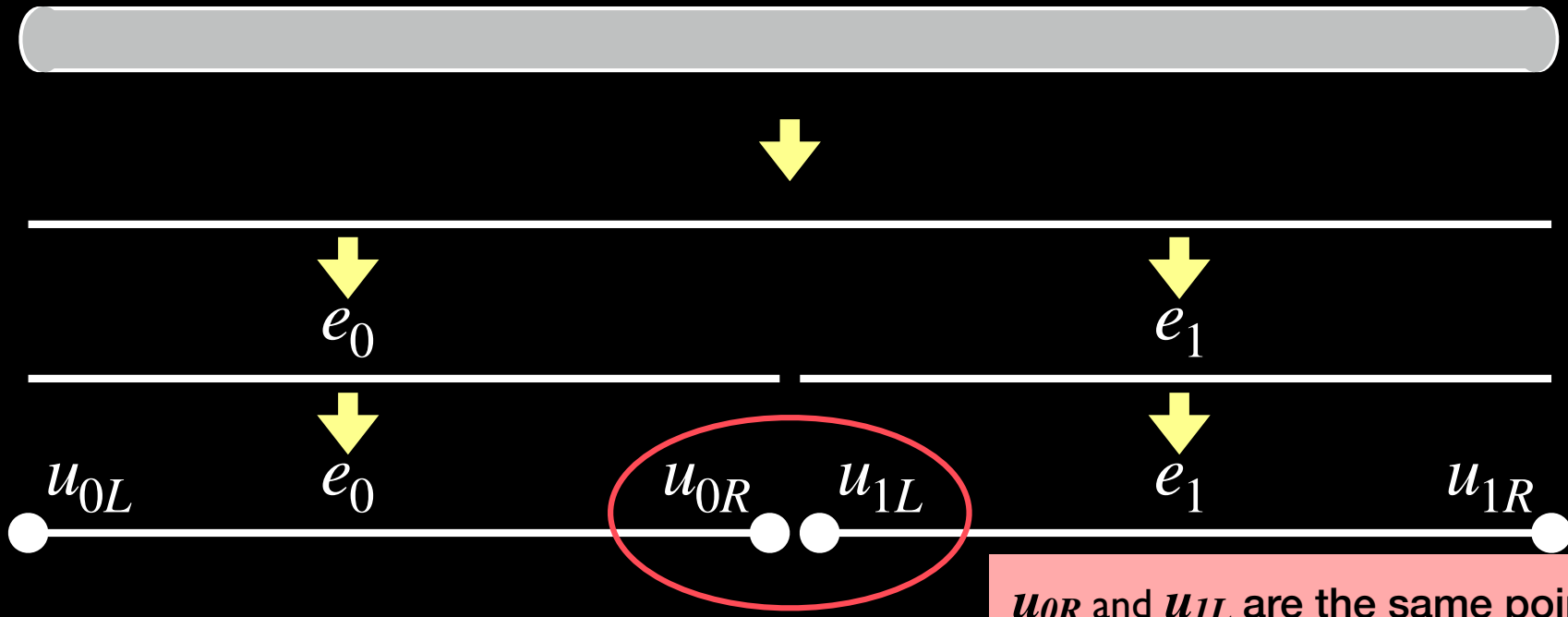
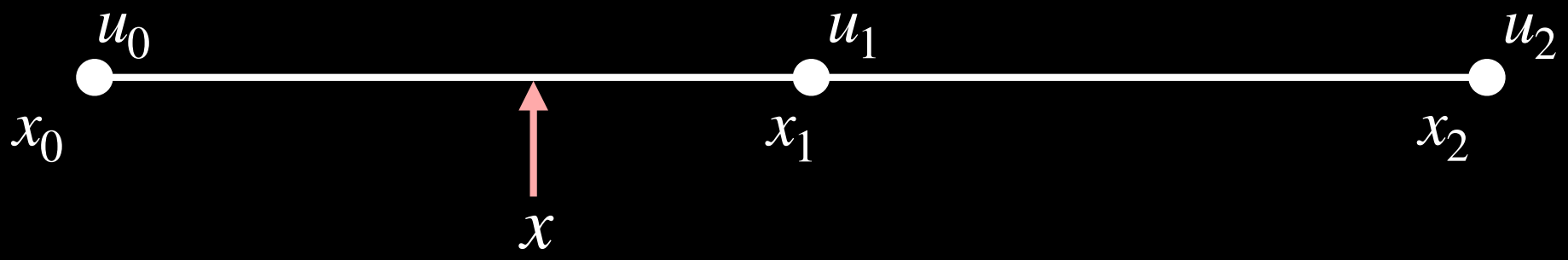


u_{0R} and u_{1L} are the same point, they define how the two segments are coupled.

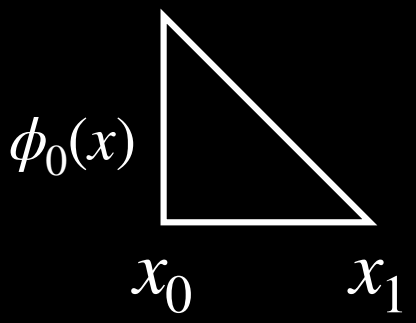
Shape Functions

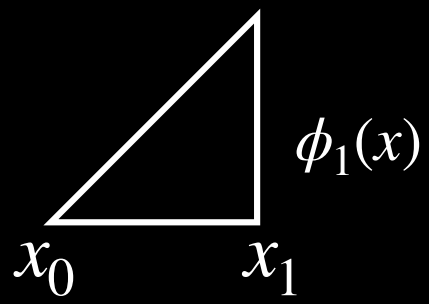
- Shape functions define how you interpolate between sample points



$$\phi_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad \phi_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$u_x = \phi_0(x)u_0 + \phi_1(x)u_1$$





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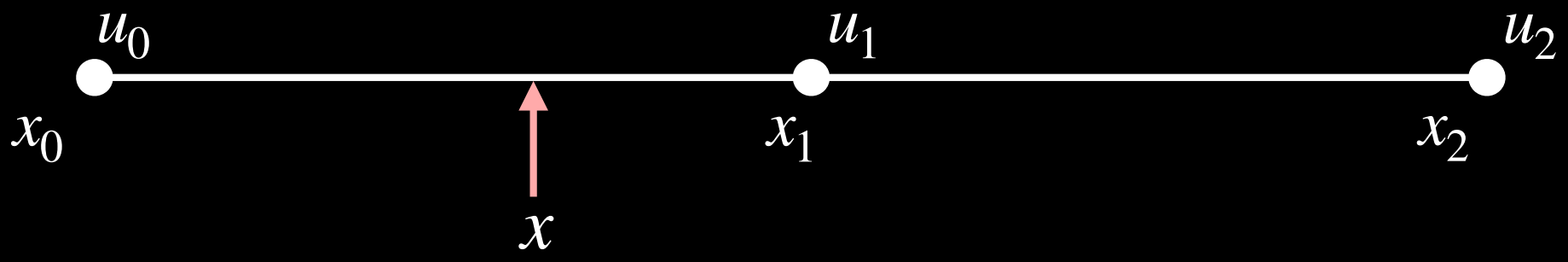

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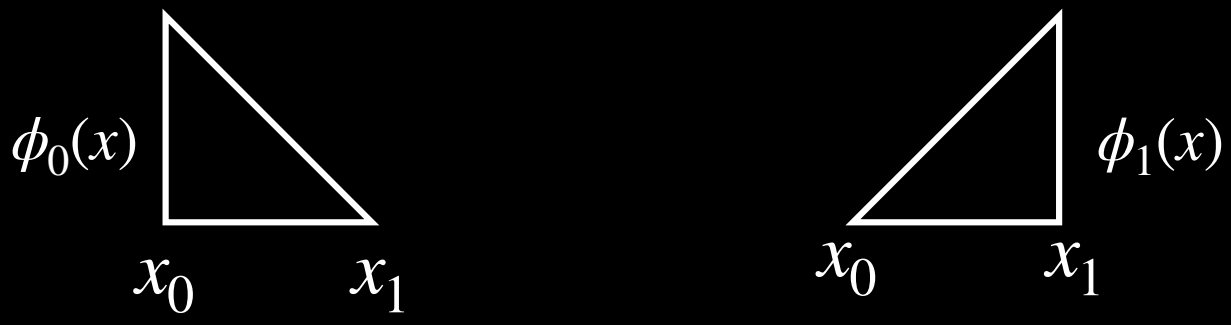


We can use this shape function to, for example, compute how a force at x is distributed at x_0 and x_1.

Combining Equations

- We have a series of equations that define the behavior of the elements, coupled at shared points.
- To determine the behavior of the entire object, combine these equations into a linear system and solve it.

k_{AB} is the coefficient for how A affects B

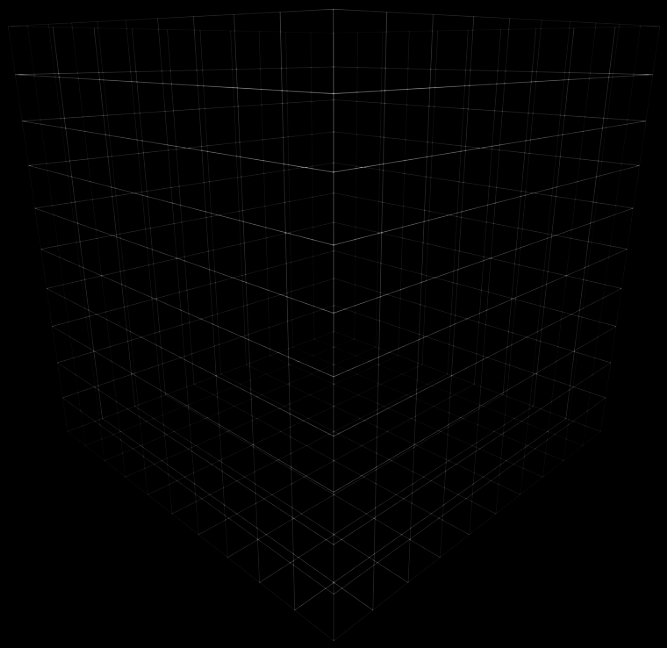
$$\begin{bmatrix} k_{00} & k_{01A} \\ k_{1A0} & k_{1A1A} \end{bmatrix} \begin{bmatrix} u_0 \\ u_{1A} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_{1A} \end{bmatrix}$$

$$\begin{bmatrix} k_{1B1B} & k_{1B2} \\ k_{21B} & k_{22} \end{bmatrix} \begin{bmatrix} u_{1B} \\ u_2 \end{bmatrix} = \begin{bmatrix} f_{1B} \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} k_{00} & k_{01A} & 0 \\ k_{1A0} & k_{1A1A} + k_{1B1B} & k_{1B2} \\ 0 & k_{21B} & k_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_{1A} + f_{1B} \\ f_2 \end{bmatrix}$$

End Result

- We have a large, sparse set of linear equations
  - NxN for N sample points
  - Each row has a number of non-zero fields equal to degree of that mesh node +1



3D grid of dimension N
Has N^3 elements
Grid point n has 7 non-zero values at

- n (itself)
- n-1, n+1 (x-axis)
- n+N, n-N (y-axis)
- $n+N_2$, $n-N^2$ (z-axis)

Often need to consider corners too: 27 non-zero values

Solving FEM

- Computing solutions for FEM problems boils down to solving these sets of equations
- Many aspects of the FEM computation scale $O(N)$, but the solver does not: $O(N^2)$, $O(N^{\frac{3}{2}})$
  - The fraction of computational time consumed by the solver increases as the model grows: it quickly dominates execution time

Two Parts

- Two basic methods: direct and indirect
  - Direct: fixed set of steps, require entire matrix in memory
  - Iterative: each step gets closer to the solution, tradeoff in speed vs. accuracy, can be distributed (used in HPC for this reason)
- Two techniques for improving solver performance
  - Factor the matrix into a format that's easier to solve
  - Laying out the data so it can be accessed quickly by the solver

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Factorization

$$\mathrm{K}\vec{u} = \vec{f}$$

Our starting problem: displacement is a function of the force and the stiffness matrix.

$$\tilde{\mathrm{L}}\tilde{\mathrm{L}}^{T}\vec{u} = \vec{f}$$

Factor stiffness matrix K into L and its transpose.
L is an upper triangular matrix, L^{T} is a lower triangular.

$$\tilde{\mathrm{L}}\vec{y} = \vec{f}$$

Solve for y. This is fast because L is an upper triangular matrix, so just work up increasing number of terms.

$$\tilde{\mathrm{L}}^{T}\vec{u} = \vec{y}$$

Solve for u. This is fast for the same reasons as solving for y.

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Key step

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Matrix Representation

- Matrices are extremely sparse: don't want to represent them completely
  - Wastes memory
  - Sparse memory access, wastes memory bandwidth
- Want to be able to access vectors of values, not just individual memory words
- Tradeoff in design
  - Only store non-zero values: vectors are short, more indirection
  - Pad values: vectors are longer, less indirection

CMNS

- Column-Major Nonzero Storage (CMNS) scheme

$$K = \begin{bmatrix} k_{11} & 0.0 & k_{13} & 0.0 & 0.0 \\ 0.0. & k_{22} & 0.0 & 0.0 & 0.0 \\ k_{31} & 0.0 & k_{33} & 0.0 & k_{35} \\ 0.0. & 0.0. & 0.0 & k_{44} & 0.0 \\ 0.0 & k_{52} & k_{53} & 0.0 & k_{55} \end{bmatrix}$$

$$\vec{K}_u^T = [k_{11}, k_{31}, k_{22}, k_{52}, k_{13}, k_{33}, k_{53}, k_{44}, k_{25}, k_{35}, k_{55}]$$

$$\vec{R}^T = [1,3,2,5,1,3,5,4,2,3,5]$$

$$\vec{L}^T = [2,2,3,1,3]$$

Count in column

Row

Coefficients

Good: dense representation
Bad: random access requires traversing *L*
Bad: fetching irregular numbers of elements

Multiplying with CMNS

$$\vec{K}_u^T = [k_{11}, k_{31}, k_{22}, k_{52}, k_{13}, k_{33}, k_{53}, k_{44}, k_{25}, k_{35}, k_{55}]$$

$$\vec{R}^T = [1,3,2,5,1,3,5,4,2,3,5]$$

$$\vec{L}^T = [2,2,3,1,3]$$

$$\vec{v} = K\vec{p}$$

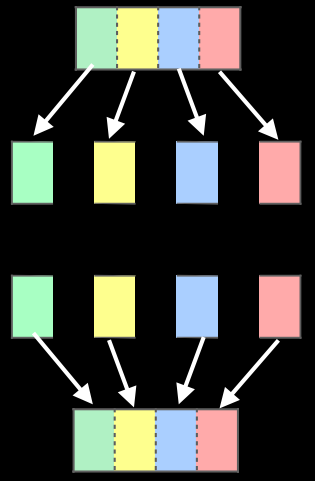
```
index = 1
for column in 1..N:
  for entry in 1..L[column]:
    v[R[index]] = V[r[index]] + K[index] * p[column]
    index++;
```

Vector Operation Costs

- Example: Cray Y-MP computer
- Fetching a vector of length V takes 19+V cycles
  - E.g., a element connected to 24 others requires V=24
  - Vector of length 24 takes 43 cycles, overhead is 44.2% of time
- High performance computing is really about *high performance:* if FPUs fall idle, you are wasting time

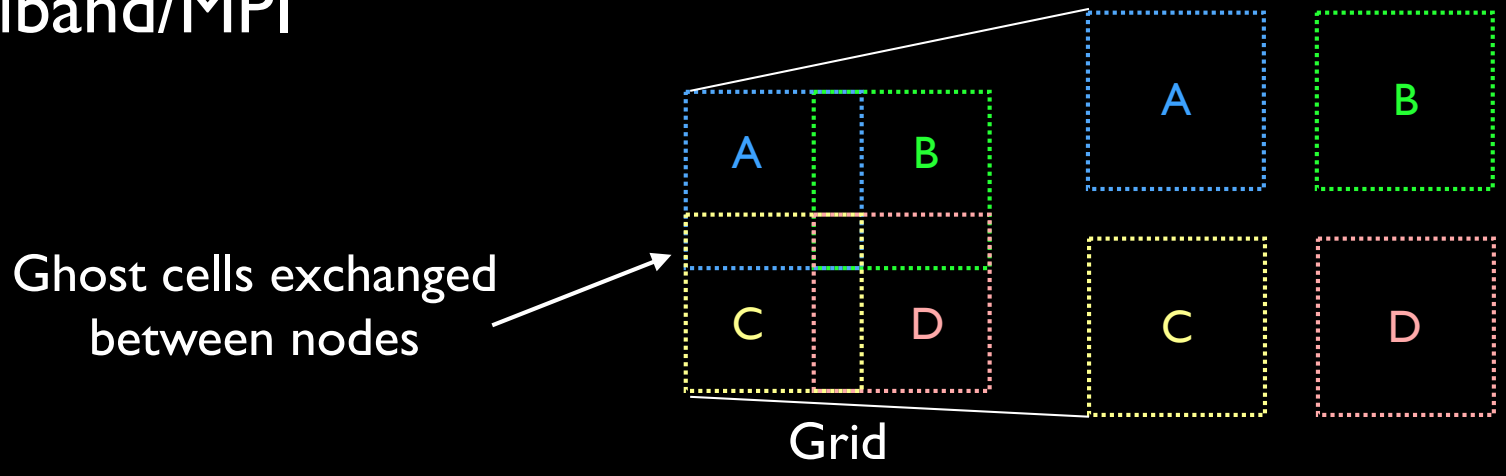
Scatter-Gather I/O

- Hardware support for transforming between sparse and dense representations
- Suppose you want to multiply a vector *v* by a sparse matrix that is represented in CMNS
  - Want to convert the dense column in CMNS into a sparse column so it can be directly multiplied by *v*
- Scatter-gather automates this movement
  - *Scatter* dense elements into a sparse representation
  - *Gather* sparse elements into a dense representation



HPC Today

- Highly parallel clusters (10,000+ cores)
- All simulations are distributed across many machines
- Bottleneck is communication and barriers rather than FPU utilization
- Ghost cells, iterative solvers
- Infiniband/MPI



Valerie Taylor's Contributions