Lecture 2: Making Sequences of Good Decisions Given a Model of the World

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CS234 Reinforcement Learning

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L2N1 Quick Check Your Understanding 1. Participation Poll

In a Markov decision process, a large discount factor γ means that short term rewards are much more influential than long term rewards. [Enter your answer in participation poll]

- True
- False
- Don't know

Question for today's lecture (not for poll): Can we construct algorithms for computing decision policies so that we can guarantee with additional computation / iterations, we monotonically improve the decision policy? Do all algorithms satisfy this property?

L2N1 Quick Check Your Understanding 1. Participation Poll

In a Markov decision process, a large discount factor γ means that short term rewards are much more influential than long term rewards. [Enter your answer in the poll]

- True
- False
- Don't know

False. A large γ implies we weigh delayed / long term rewards more. $\gamma = 0$ only values immediate rewards

Question for today's lecture (not for poll): Can we construct algorithms for computing decision policies so that we can guarantee with additional computation / iterations, we monotonically improve the decision policy? Do all algorithms satisfy this property?
Yes it is possible! We will see this today. Not all of them do.

Class Tasks and Updates

- Homework 1 out shortly. Due Friday April 12 at 6pm.
- Office hours will start Friday. See Ed for days, times of group and 1:1 office hours, and we will update the calendar on the website with locations shortly.

Today's Plan

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

Today: Given a model of the world

dynamics $p(s'|s,a)$
reward model

- Markov Processes (last time)
- Markov Reward Processes (MRPs) (continue from last time)
- Markov Decision Processes (MDPs)
- Evaluation and Control in MDPs

Return & Value Function

- Definition of Horizon (H)
  - Number of time steps in each episode
  - Can be infinite
  - Otherwise called **finite** Markov reward process
- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon H

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1}$$

- Definition of State Value Function, $V(s)$ (for a MRP)
  - Expected return from starting in state s

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1} | s_t = s]$$

Computing the Value of an Infinite Horizon Markov Reward Process

- Markov property provides structure
- MRP value function satisfies

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s)V(s')}_{\text{Discounted sum of future rewards}}$$

Matrix Form of Bellman Equation for MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$
\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \overbrace{\begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix}}^{P} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}
$$

$$V = R + \gamma P V$$

$$V - \gamma P V = R$$

$$V(I - \gamma P) = R \qquad I = \text{identity matrix}$$

$$V = (I - \gamma P)^{-1} R$$

Analytic Solution for Value of MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$\begin{aligned} V &= R + \gamma P V \\ V - \gamma P V &= R \\ (I - \gamma P) V &= R \\ V &= (I - \gamma P)^{-1} R \end{aligned}$$

- Solving directly requires taking a matrix inverse $\sim O(N^3)$
- Requires that $(I - \gamma P)$ is invertible

Iterative Algorithm for Computing Value of a MRP

- Dynamic programming
- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence

iteration

  - For all s in S

$$V_k(s) = R(s) + \gamma \sum_{s' \in S} P(s'|s) V_{k-1}(s')$$

- Computational complexity: $O(|S|^2)$ for each iteration ($|S| = N$)

Markov Decision Process (MDP)

- Markov Decision Process is Markov Reward Process + actions
- Definition of MDP
  - S is a (finite) set of Markov states $s \in S$
  - A is a (finite) set of actions $a \in A$
  - P is dynamics/transition model for **each action**, that specifies $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - R is a reward function[1]

$$R(s_t = s, a_t = a) = \mathbb{E}[r_t | s_t = s, a_t = a]$$

  - Discount factor $\gamma \in [0, 1]$
- MDP is a tuple: (S, A, P, R, γ)

[1] Reward is sometimes defined as a function of the current state, or as a function of the (state, action, next state) tuple. Most frequently in this class, we will assume reward is a function of state and action

Example: Mars Rover MDP

s_1	s_2	s_3	s_4	s_5	s_6	s_7
+1	0	0	 0	0	0	+10

$$P(s'|s,a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P(s'|s,a_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 2 deterministic actions

MDP Policies

- Policy specifies what action to take in each state
  - Can be deterministic or stochastic
- For generality, consider as a conditional distribution
  - Given a state, specifies a distribution over actions
- Policy: $\pi(a|s) = P(a_t = a|s_t = s)$

MDP + Policy

- MDP + $\pi(a|s)$ = Markov Reward Process
- Precisely, it is the MRP $(S, R^\pi, P^\pi, \gamma)$, where

$$R^\pi(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$
$$P^\pi(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

- Implies we can use same techniques to evaluate the value of a policy for a MDP as we could to compute the value of a MRP, by defining a MRP with R^π and P^π

MDP Policy Evaluation, Iterative Algorithm

- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = \sum_a \pi(a|s) \left[R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) V_{k-1}^\pi(s') \right]$$

- This is a **Bellman backup** for a particular policy
- Note that if the policy is deterministic then the above update simplifies to

$$V_k^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

Exercise L2E1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, $\ldots$
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \; \forall s$, assume $V_k =$[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
- Compute $V_{k+1}(s_6)$

See answer at the end of the slide deck. If you'd like practice, work this out and then check your answers.

Check Your Understanding Poll L2N2

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

$|S|$

$|A|$

- We will shortly be interested in not just evaluating the value of a single policy, but finding an optimal policy. Given this it is informative to think about properties of the potential policy space.
- First for the Mars rover example [7 discrete states (location of rover); 2 actions: Left or Right]
- How many deterministic policies are there?
- Select answer on the participation poll: 2 / 14 / 7^2 / 2^7 / Not sure
- Is the optimal policy (one with highest value) for a MDP unique?
- Select answer on the participation poll: Yes / No / Not sure

Check Your Understanding L2N2

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

- 7 discrete states (location of rover)
- 2 actions: Left or Right
- How many deterministic policies are there?
  2^7
- Is the highest reward policy for a MDP always unique?
  No, there may be two policies with the same (maximal) value function.

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There **exists a unique optimal value function**
- Optimal policy for a MDP in an infinite horizon problem is deterministic

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There exists a unique optimal value function
- Optimal policy for a MDP in an infinite horizon problem (agent acts forever is
  - Deterministic
  - Stationary (does not depend on time step)
  - Unique? Not necessarily, may have two policies with identical (optimal) values

Policy Search

- One option is searching to compute best policy
- Number of deterministic policies is $|A|^{|S|}$
- Policy iteration is generally more efficient than enumeration

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

New Definition: State-Action Value Q

- State-action value of a policy

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi}(s')$$

- Take action a, then follow the policy π

Policy Improvement

- Compute state-action value of a policy π_i
  - For s in S and a in A:

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

- Compute new policy π_{i+1}, for all $s \in S$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a) \; \forall s \in S$$

if $\pi_i = \pi_{i+1}$ is $Q^{\pi_i} = Q^{\pi_{i+1}}$

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

Q

$\pi_{i+1}(s) = \arg\max_a Q(s, a)$

Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

Delving Deeper Into Policy Improvement Step

$$\underline{Q^{\pi_i}}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

policy eval step

$Q^{\pi_i}(s, \pi_i(s))$

$$\underbrace{\max_a \underline{Q^{\pi_i}}(s,a)}_{\geq Q^{\pi_i}(s,a)\ \forall a} \geq R(s,\pi_i(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_i(s)) V^{\pi_i}(s') = \underline{V^{\pi_i}}(s)$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

- Suppose we take $\pi_{i+1}(s)$ for one action, then follow π_i forever
  - Our expected sum of rewards is at least as good as if we had always followed π_i
- But new proposed policy is to always follow π_{i+1} ...

Monotonic Improvement in Policy

- Definition
$$V^{\pi_1} \geq V^{\pi_2} : V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s \in S$$
- Proposition: $V^{\pi_{i+1}} \geq V^{\pi_i}$ with strict inequality if π_i is suboptimal, where π_{i+1} is the new policy we get from policy improvement on π_i

Proof: Monotonic Improvement in Policy

definition of $\max_a Q^{\pi_i}$

$$
\begin{aligned}
V^{\pi_i}(s) &\le \max_a Q^{\pi_i}(s,a) \\
&= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')
\end{aligned}
$$

$$= R(s, \pi_{i+1}(s)) + \gamma \sum_{s'} P(s'|s, \pi_{i+1}(s)) V^{\pi_i}(s')$$

$$\le R(s, \pi_{i+1}(s)) + \gamma \sum_{s'} P(s'|s, \pi_{i+1}(s)) \max_{a'} Q^{\pi_i}(s', a')$$

$$= R(s, \pi_{i+1}(s)) + \gamma \sum_{s'} P(s'|s, \pi_{i+1}(s)) \cdot \left[R(s', \pi_{i+1}(s')) + \gamma \sum_{s''} P(s''|s', \pi_{i+1}(s')) V^{\pi_i}(s'') \right]$$

$$\ldots$$

$$= V^{\pi_{i+1}}(s)$$

Proof: Monotonic Improvement in Policy

$$
\begin{aligned}
V^{\pi_i}(s) \le & \max_a Q^{\pi_i}(s,a) \\
= & \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s') \\
= & R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) V^{\pi_i}(s') \text{ //by the definition of } \pi_{i+1} \\
\le & R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) \left(\max_{a'} Q^{\pi_i}(s', a') \right) \\
= & R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) \\
& \left(R(s', \pi_{i+1}(s')) + \gamma \sum_{s'' \in S} P(s''|s', \pi_{i+1}(s')) V^{\pi_i}(s'') \right) \\
& \vdots \\
= & V^{\pi_{i+1}}(s)
\end{aligned}
$$

Check Your Understanding L2N3: Policy Iteration (PI)

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

We will talk about ties in a minute!

- **If policy doesn't change, can it ever change again?**
- Select on participation poll: Yes / No / Not sure
- **Is there a maximum number of iterations of policy iteration?**
- Select on participation poll: Yes / No / Not sure

Lecture Break after Policy Iteration

Results for Check Your Understanding L2N3 Policy Iteration

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$
- **If policy doesn't change, can it ever change again?**
  No (if π^* is unique $\forall s$)
- **Is there a maximum number of iterations of policy iteration?**
  $|A|^{|S|}$ since that is the maximum number of policies, and as the policy improvement step is monotonically improving, each policy can only appear in one round of policy iteration unless it is an optimal policy.

Check Your Understanding Explanation of Policy Not Changing

- Suppose for all $s \in S$, $\pi_{i+1}(s) = \pi_i(s)$
- Then for all $s \in S$, $Q^{\pi_{i+1}}(s, a) = Q^{\pi_i}(s, a)$
- Recall policy improvement step

$$Q^{\pi_i}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi_i}(s')$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

$$\pi_{i+2}(s) = \arg\max_a Q^{\pi_{i+1}}(s, a) = \arg\max_a Q^{\pi_i}(s, a)$$

Therefore policy cannot ever change again

(assuming π^* is unique per s)

MDP: Computing Optimal Policy and Optimal Value

- Policy iteration computes infinite horizon value of a policy and then improves that policy
- Value iteration is another technique
  - Idea: Maintain optimal value of starting in a state s if have a finite number of steps k left in the episode
  - Iterate to consider longer and longer episodes

Bellman Equation and Bellman Backup Operators

- Value function of a policy must satisfy the Bellman equation

$$V^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V^\pi(s')$$

- Bellman backup operator
  - Applied to a value function
  - Returns a new value function
  - Improves the value if possible

$$BV(s) = \max_a \left[R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) V(s') \right]$$

  - BV yields a value function over all states s

Value Iteration (VI)

L_∞ norm

$= \max_s | V_{k+1}(s) - V_k(s) |$

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until convergence: (for ex. $||V_{k+1} - V_k||_\infty \leq \epsilon$)
  - For each state s

$$V_{k+1}(s) = \max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

  - View as Bellman backup on value function

$V = 0 \ \forall s$

$V_{k+1}(\cancel{t}s) = \max_a r(s,a)$

$$V_{k+1} = BV_k$$

$$\pi_{k+1}(s) = \arg\max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V(s')$$

- Policy evaluation amounts to computing the fixed point of B^π
- To do policy evaluation, repeatedly apply operator until V stops changing

$$V^\pi = B^\pi B^\pi \cdots B^\pi V$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V(s')$$

- To do policy improvement

$Q^{\pi_k}(s,a)$

$$\pi_{k+1}(s) = \underline{\arg} \max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a)V^{\pi_k}(s') \right]$$

Going Back to Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until convergence: (for ex. $||V_{k+1} - V_k||_\infty \leq \epsilon$)
  - For each state s

$$V_{k+1}(s) = \max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

  - Equivalently, in Bellman backup notation

$$V_{k+1} = BV_k$$

- To extract optimal policy if can act for $k+1$ more steps,

$$\pi(s) = \arg\max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_{k+1}(s') \right]$$

Contraction Operator

- Let O be an operator,and $|x|$ denote (any) norm of x
- If $|OV - OV'| \leq |V - V'|$, then O is a contraction operator

Will Value Iteration Converge?

- Yes, if discount factor $\gamma < 1$, or end up in a terminal state with probability 1
- Bellman backup is a contraction if discount factor, $\gamma < 1$
- If apply it to two different value functions, distance between value functions shrinks after applying Bellman equation to each

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$\|BV_k - BV_j\| = \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\|$$

$$\leq \max_a \left\| \cancel{R(s,a)} + \gamma \sum_{s'} P(s'|s,a) V_k(s') - \cancel{R(s,a)} - \gamma \sum_{s'} P(s'|s,a) V_j(s') \right\|$$

$$= \max_a \gamma \left\| \sum_{s'} P(s'|s,a) (V_k(s') - V_j(s')) \right\|$$

$$\leq \max_a \gamma \left\| \sum_{s'} P(s'|s,a) \|V_k - V_j\| \right\|$$

$$= \max_a \gamma \|V_k - V_j\| \, \| \underbrace{\textstyle\sum_{s'} P(s'|s,a)}_{=1} \|$$

$$= \cancel{\max} \, \gamma \|V_k - V_j\|$$

$V_0 \quad V_1 \quad V_2 \quad V_3 \quad V_4 \quad \ldots$

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

$Q_j(s, a_1)$ or $Q_j(s, a)$

a_1 or another a $a \neq a_1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

a_1

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

- Note: Even if all inequalities are equalities, this is still a contraction if $\gamma < 1$

Opportunities for Out-of-Class Practice

- Prove value iteration converges to a unique solution for discrete state and action spaces with $\gamma < 1$
- Does the initialization of values in value iteration impact anything?
- Is the value of the policy extracted from value iteration at each round guaranteed to monotically improve (if executed in the real infinite horizon problem), like policy iteartion?

Value Iteration for Finite Horizon H

V_k = optimal value if making k more decisions
π_k = optimal policy if making k more decisions

- Initialize $V_0(s) = 0$ for all states s
- For $k = 1 : H$
  - For each state s

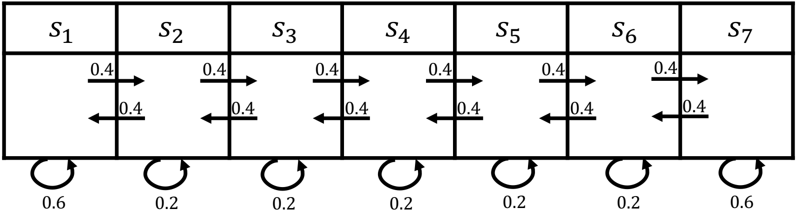
$$V_{k+1}(s) = \max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

$$\pi_{k+1}(s) = \arg\max_a \left[R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

Computing the Value of a Policy in a Finite Horizon

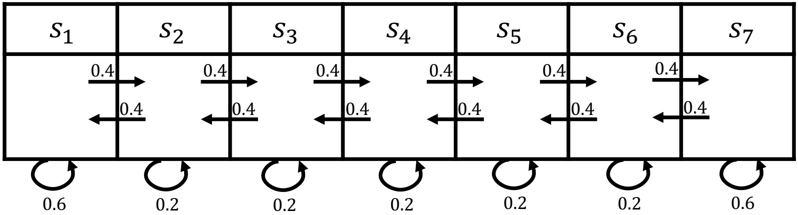
- Alternatively can estimate by simulation
  - Generate a large number of episodes
  - Average returns
  - Concentration inequalities bound how quickly average concentrates to expected value
  - Requires **no assumption** of Markov structure

Example: Mars Rover



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step (H=4) episodes, $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$

Example: Mars Rover



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step (H=4) episodes, start state s_4 , $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$
  - s_4, s_4, s_5, s_4: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 0 = 0$
  - s_4, s_3, s_2, s_1: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 1 = 0.125$

Question: Finite Horizon Policies

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until $k == H$:
    - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks?

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Is optimal policy stationary (independent of time step) in finite horizon tasks?
In general no.

Value vs Policy Iteration

- Value iteration:
  - Compute optimal value for horizon $= k$
    - Note this can be used to compute optimal policy if horizon $= k$
  - Increment k
- Policy iteration
  - Compute infinite horizon value of a policy
  - Use to select another (better) policy
  - Closely related to a very popular method in RL: policy gradient

RL Terminology: Models, Policies, Values

- **Model**: Mathematical models of dynamics and reward
- **Policy**: Function mapping states to actions
- **Value function**: future rewards from being in a state and/or action when following a particular policy

What You Should Know

- Define MP, MRP, MDP, Bellman operator, contraction, model, Q-value, policy
- Be able to implement
  - Value Iteration
  - Policy Iteration
- Give pros and cons of different policy evaluation approaches
- Be able to prove contraction properties
- Limitations of presented approaches and Markov assumptions
  - Which policy evaluation methods require the Markov assumption?

Where We Are

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

Exercise L2E1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example, Answer

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, ...
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \ \forall s$, assume $V_k =$[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
- Compute $V_{k+1}(s_6)$

$$\begin{aligned} V_{k+1}(s_6) &= r(s_6) + \gamma \sum_{s'} p(s'|s_6, a_1) V_k(s') && (1) \\ &= 0 + 0.5 * (0.5 * 10 + 0.5 * 0) && (2) \\ &= 2.5 && (3) \end{aligned}$$

Check Your Understanding L2N1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

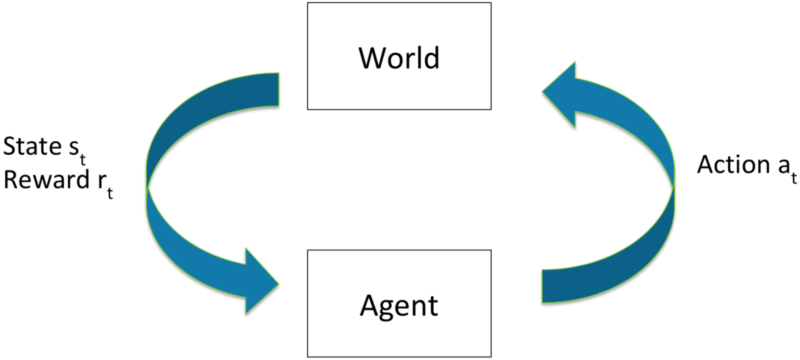
- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, $\ldots$
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \; \forall s$, assume $V_k =$[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
-
$$V_k^{\pi}(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^{\pi}(s')$$

$$V_{k+1}(s_6) = r(s_6, a_1) + \gamma * 0.5 * V_k(s_6) + \gamma * 0.5 * V_k(s_7)$$

$$V_{k+1}(s_6) = 0 + 0.5 * 0.5 * 0 + .5 * 0.5 * 10$$

$$V_{k+1}(s_6) = 2.5$$

Full Observability: Markov Decision Process (MDP)



- MDPs can model a huge number of interesting problems and settings
  - Bandits: single state MDP
  - Optimal control mostly about continuous-state MDPs
  - Partially observable MDPs = MDP where state is history

Recall: Markov Property

- Information state: sufficient statistic of history
- State s_t is Markov if and only if:

$$p(s_{t+1}|s_t, a_t) = p(s_{t+1}|h_t, a_t)$$

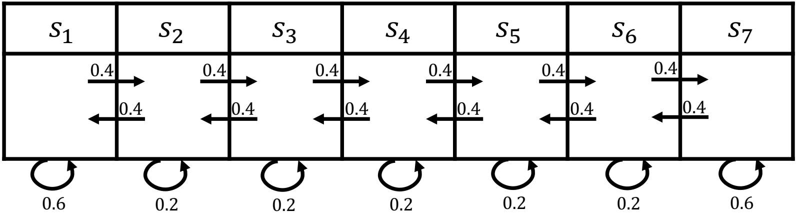
- Future is independent of past given present

Markov Process or Markov Chain

- Memoryless random process
  - Sequence of random states with Markov property
- Definition of Markov Process
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $p(s_{t+1} = s' | s_t = s)$
- Note: no rewards, no actions
- If finite number (N) of states, can express P as a matrix

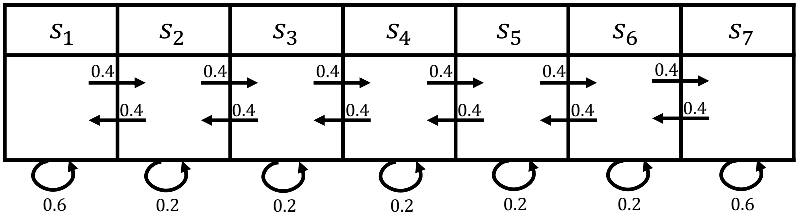
$$P = \begin{pmatrix} P(s_1|s_1) & P(s_2|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & P(s_2|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \vdots & \ddots & \vdots \\ P(s_1|s_N) & P(s_2|s_N) & \cdots & P(s_N|s_N) \end{pmatrix}$$

Example: Mars Rover Markov Chain Transition Matrix, P



$$P = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

Example: Mars Rover Markov Chain Episodes



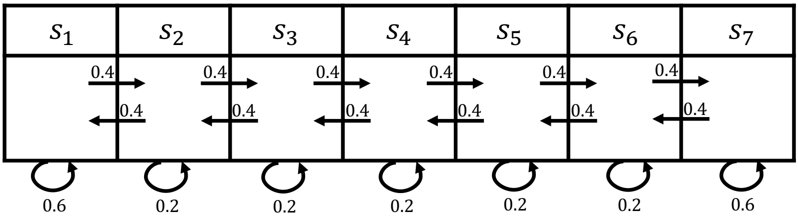
Example: Sample episodes starting from S4

- $s_4, s_5, s_6, s_7, s_7, s_7, \ldots$
- $s_4, s_4, s_5, s_4, s_5, s_6, \ldots$
- $s_4, s_3, s_2, s_1, \ldots$

Markov Reward Process (MRP)

- Markov Reward Process is a Markov Chain + rewards
- Definition of Markov Reward Process (MRP)
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $P(s_{t+1} = s' | s_t = s)$
  - R is a reward function $R(s_t = s) = \mathbb{E}[r_t | s_t = s]$
  - Discount factor $\gamma \in [0, 1]$
- Note: no actions
- If finite number (N) of states, can express R as a vector

Example: Mars Rover MRP



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states