EXTENSIONS OF LIPSCHITZ MAPPINGS INTO A HILBERT SPACE

William B. Johnson\(^1\) and Joram Lindenstrauss\(^2\)

INTRODUCTION

In this note we consider the following extension problem for Lipschitz functions: Given a metric space \(X\) and \(n = 2, 3, 4, \ldots\), estimate the smallest constant \(L = L(X, n)\) so that every mapping \(f\) from every \(n\)-element subset of \(X\) into \(\ell_2\) extends to a mapping \(\tilde{f}\) from \(X\) into \(\ell_2\) with

\[
\|\tilde{f}\|_{\ell_2} \leq L \|f\|_{\ell_2p}.
\]

(Here \(\|g\|_{\ell_2p}\) is the Lipschitz constant of the function \(g\).) A classical result of Kirszbraun's [14, p. 48] states that \(L(\ell_2, n) = 1\) for all \(n\), but it is easy to see that \(L(X, n) \to \infty\) as \(n \to \infty\) for many metric spaces \(X\).

Marcus and Pisier [10] initiated the study of \(L(X, n)\) for \(X = L_p\). (For brevity, we will use hereafter the notation \(L(p, n)\) for \(L(L_p(0,1), n)\).) They prove that for each \(1 < p < 2\) there is a constant \(C(p)\) so that for \(n = 2, 3, 4, \ldots\),

\[
L(p, n) \leq C(p) (\log n)^{1/p - 1/2}.
\]

The main result of this note is a verification of their conjecture that for some constant \(C\) and all \(n = 2, 3, 4, \ldots\),

\[
L(X, n) \leq C(\log n)^{1/2}
\]

for all metric spaces \(X\). While our proof is completely different from that of Marcus and Pisier, there is a common theme: Probabilistic techniques developed for linear theory are combined with Kirszbraun's theorem to yield extension theorems.

The main tool for proving Theorem 1 is a simply stated elementary geometric lemma, which we now describe: Given \(n\) points in Euclidean space, what

\(^1\)Supported in part by NSF MCS-7903042.

\(^2\)Supported in part by NSF MCS-8102714.
is the smallest \( k = k(n) \) so that these points can be moved into \( k \)-dimensional Euclidean space via a transformation which expands or contracts all pairwise distances by a factor of at most \( 1 + \varepsilon \)? The answer, that \( k \leq C(\varepsilon) \log n \), is a simple consequence of the isoperimetric inequality for the \( n \)-sphere in the form studied in [2].

It seems likely that the Marcus-Pisier result and Theorem 1 give the right order of growth for \( L(p, n) \). While we cannot verify this, in Theorem 3 we get the estimate

\[
L(p, n) \geq \delta \left( \frac{\log n}{\log \log n} \right)^{1/p - 1/2} (1 \leq p < 2)
\]

for some absolute constant \( \delta > 0 \). (Throughout this paper we use the convention that \( \log x \) denotes the maximum of \( 1 \) and the natural logarithm of \( x \).) This of course gives a lower estimate of

\[
\delta \left( \frac{\log n}{\log \log n} \right)^{1/2}
\]

for \( L(\infty, n) \). That our approach cannot give a lower bound of \( \delta (\log n)^{1/p - 1/2} \) for \( L(p, n) \) is shown by Theorem 2, which is an extension theorem for mappings into \( \ell_2 \) whose domains are \( \varepsilon \)-separated.

The minimal notation we use is introduced as needed. Here we note only that \( B_Y(y, \varepsilon) \) (respectively, \( b_Y(y, \varepsilon) \)) is the closed (respectively, open) ball in \( Y \) about \( y \) of radius \( \varepsilon \). If \( y = 0 \), we use \( B_Y(\varepsilon) \) and \( b_Y(\varepsilon) \), and we drop the subscript \( Y \) when there is no ambiguity. \( S(Y) \) is the unit sphere of the normed space \( Y \). For isomorphic normed spaces \( X \) and \( Y \), we let

\[
d(X,Y) = \inf \|T\| \|T^{-1}\|,
\]

where the inf is over all invertible linear operators from \( X \) onto \( Y \). Given a bounded Banach space valued function \( f \) on a set \( K \), we set

\[
\|f\|_\infty = \sup_{x \in K} \|f(x)\|.
\]

1. THE EXTENSION THEOREMS

We begin with the geometrical lemma mentioned in the introduction.

**Lemma 1.** For each \( 1 > \tau > 0 \) there is a constant \( K = K(\tau) > 0 \) so that if \( A \subseteq \ell_2^n \), \( A = n \) for some \( n = 2, 3, \ldots \), then there is a mapping \( f \) from \( A \) onto a subset of \( \ell_2^k \) \((k \leq [K \log n])\) which satisfies
PROOF. The proof will show that if one chooses at random a rank \( k \) orthogonal projection on \( \ell^n_2 \), then, with positive probability (which can be made arbitrarily close to one by adjusting \( k \)), the projection restricted to \( A \) will satisfy the condition on \( \tilde{f} \). To make this precise, we let \( Q \) be the projection onto the first \( k \) coordinates of \( \ell^n_2 \) and let \( \sigma \) be normalized Haar measure on \( O(n) \), the orthogonal group on \( \ell^n_2 \). Then the random variable

\[
\tilde{f} : (O(n), \sigma) \to L(\ell^n_2)
\]

defined by

\[
f(u) = U^* QU
\]

determines the notion of "random rank \( k \) projection." The applications of Levy's inequality in the first few self-contained pages of [2] make it easy to check that \( f(u) \) has the desired property. For the convenience of the reader, we follow the notation of [2].

Let \( \|\cdot\| \) denote the usual Euclidean norm on \( \mathbb{R}^n \) and for \( 1 \leq k \leq n \) and \( x \in \mathbb{R}^n \) set

\[
r(x) = r_k(x) = \sqrt{n} \left( \sum_{i=1}^{k} x(i)^2 \right)^{1/2},
\]

which is equal to

\[
\sqrt{n} \|Qx\|
\]

for our eventual choice of \( k = \lfloor K \log n \rfloor \). Thus \( r(\cdot) \) is a semi-norm on \( \ell^n_2 \) which satisfies

\[
r(x) \leq \sqrt{n} \|x\| \quad (x \in \ell^n_2).
\]

(In [2], \( r(\cdot) \) is assumed to be a norm, but inasmuch as the left estimate \( a \|x\| \leq r(x) \) in formula (2.5) of [2] is not needed in the present situation, it is okay that \( r(\cdot) \) is only a semi-norm.)

Setting

\[
B = \left\{ \frac{x - y}{\|x - y\|} : x, y \in A; x \neq y \right\} \subset S^{n-1},
\]

we want to select \( U \in O(n) \) so that for some constant \( M \),
Let $M_r$ be the median of $r(\cdot)$ on $S^{n-1}$, so that
\[ \mu_{n-1} [x \in S^{n-1} : r(x) \geq M_r] \geq 1/2 \]
and
\[ \mu_{n-1} [x \in S^{n-1} : r(x) \leq M_r] \leq 1/2 \]
where $\mu_{n-1}$ is normalized rotationally invariant measure on $S^{n-1}$.

We have from page 58 of [2] that for each $y \in S^{n-1}$ and $\varepsilon > 0$,
\[ \sigma[U \in O(n) : M_r - \sqrt{n} \varepsilon \leq r(Uy) \leq M_r + \sqrt{n} \varepsilon] \geq 1 - 4 \exp \left( \frac{-n \varepsilon^2}{2} \right). \]
Hence
\[ \sigma[U \in O(n) : M_r - \sqrt{n} \varepsilon \leq r(Uy) \leq M_r + \sqrt{n} \varepsilon \text{ for all } y \in B] \geq 1 - 2n(n+1) \exp \left( \frac{-n \varepsilon^2}{2} \right). \]

By Lemma 1.7 of [2], there is a constant
\[ C \leq 4 \sum_{m=1}^{\infty} (m+1) e^{-m/2} \]
so that
\[ \left| \int_{S^{n-1}} r(x) \, d\mu_{n-1}(x) - M_r \right| < C. \]

We now repeat a known argument for estimating $\int_{S^{n-1}} r(x) \, d\mu_{n-1}(x)$ which uses only Khintchine's inequality.

For $1 \leq k \leq n$ we have:
\[ \frac{1}{k} \sum_{i=1}^{k} x(i) \left| \int_{S^{n-1}} \delta_1 \, d\mu_{n-1}(x) \right| = \]
\[ = \sqrt{k} \int_{S^{n-1}} \left| < x, \sum_{i=1}^{k} \delta_1 > \right| \, d\mu_{n-1}(x) \]
\[ \leq \sqrt{n} \int_{S^{n-1}} \left| < x, \delta_1 > \right| \, d\mu_{n-1}(x) \]
[by the rotational invariance of $\mu_{n-1}$].

Setting
\[ a_n = \int_{S^{n-1}} \left| < x, \delta_1 > \right| \, d\mu_{n-1}(x), \]
we have from Khintchine's inequality that for each $1 \leq k \leq n$,
\[
\sqrt{n_k} a_k \leq \int_{S^{n-1}} r_k(x) \, d\mu_{n-1}(x) \leq \sqrt{2n_k} a_k.
\]
(We plugged in the exact constant of $\sqrt{2}$ in Khintchine's inequality calculated in [5] and [13], but of course any constant would serve as well.)

Since obviously $r_n(x) = \sqrt{n}$, we conclude that for $1 \leq k \leq n$
\[
(1.3) \quad \sqrt{k/3} \leq \int_{S^{n-1}} r_k(x) \, d\mu_{n-1}(x) \leq \sqrt{k}.
\]

Specializing now to the case $k = \lfloor K \log n \rfloor$, we have from (1.2) and (1.3) that
\[
\sqrt{k/3} \leq M_r
\]
at least for $K \log n$ sufficiently large. Thus if we define
\[
\varepsilon = \tau \sqrt{k/3n}
\]
we get from (1.1) that
\[
\sigma \left[ U \in O(n) : (1 - \tau)M_r \leq \tau(Uy) \leq (1 + \tau)M_r \quad \text{for all } y \in B \right] \\
\geq 1 - 2n(n + 1) \exp \left( -\frac{\tau^2 k}{18} \right) \\
\geq 1 - 2n(n + 1) \exp \left( -\frac{\tau^2 K \log n}{18} \right)
\]
which is positive if, say,
\[
K \geq \left( \frac{10}{\tau} \right)^2.
\]

It is easily seen that the estimate $K \log n$ in Lemma 1 cannot be improved. Indeed, in a ball of radius 2 in $\ell^k_2$ there are at most $4^k$ vectors $\{x_i\}$ so that $\|x_i - x_j\| \geq 1$ for every $i \neq j$ (see the proof of Lemma 3 below). Hence for $\tau$ sufficiently small there is no map $F$ which maps an orthonormal set with more than $4^k$ vectors into a $k$-dimensional subspace of $\ell^2_2$ with
\[
\|F\|_{\ell^p} \|F^{-1}\|_{\ell^p} \leq \frac{1 + \tau}{1 - \tau}.
\]

We can now verify the conjecture of Marcus and Pisier [10].
THEOREM 1. \[\sup_{n=2, 3, \ldots} (\log n)^{-1/2} \text{L}(\infty, n) < \infty.\] In other words: there is a constant \(K\) so that for all metric spaces \(X\) and all finite subsets \(M\) of \(X\) (\(\text{card}\ M = n\), say) every function \(f\) from \(M\) into \(\ell_2\) has a Lipschitz extension \(\tilde{f}: X \rightarrow \ell_2\) which satisfies
\[\|\tilde{f}\|_{\ell_1p} \leq K \sqrt{\log n} \|f\|_{\ell_1p}.\]

PROOF. Given \(X, M \subset X\) with \(\text{card}\ M = n\), and \(f: M \rightarrow \ell_2\), set \(A = f[M]\).
We apply Lemma 1 with \(\tau = 1/2\) to get a one-to-one function \(g^{-1}\) from \(A\) onto a subset \(g^{-1}[A]\) of \(\ell_2^k\) (where \(k \leq K \log n\)) which satisfies
\[\|g^{-1}\|_{\ell_1p} \leq 1; \quad \|g\|_{\ell_1p} \leq 3.\]

By Kirszbraun's theorem, we can extend \(g\) to a function \(\tilde{g}: \ell_2^k \rightarrow \ell_2\) in such a way that
\[\|\tilde{g}\|_{\ell_1p} \leq 3.\]

Let \(I: \ell_2^k \rightarrow \ell_\infty^k\) denote the formal identity map, so that
\[\|I\| = 1, \quad \|I^{-1}\| = \sqrt{k}.\]

Then
\[h = Ig^{-1}f, \quad h: M \rightarrow \ell_\infty^k\]
has Lipschitz norm at most \(\|f\|_{\ell_1p}\), so by the non-linear Hahn-Banach theorem (see, e.g., p. 48 of [14]), \(h\) can be extended to a mapping
\[\tilde{h}: X \rightarrow \ell_\infty^k\]
which satisfies
\[\|\tilde{h}\|_{\ell_1p} \leq \|f\|_{\ell_1p}.\]

Then
\[\tilde{f} = \tilde{g} I^{-1} \tilde{h}; \quad \tilde{f}: X \rightarrow \ell_2\]
is an extension of \(f\) and satisfies
\[\|\tilde{f}\|_{\ell_1p} \leq 3 \sqrt{k} \|f\|_{\ell_1p} \leq 3K \sqrt{\log n} \|f\|_{\ell_1p}.\]
Next we outline our approach to the problem of obtaining a lower bound for $L(\infty,n)$. Take for $f$ the inclusion mapping from an $\epsilon$-net for $S^{N-1}$ into $\ell^N_2$, and consider $\ell^N_2$ isometrically embedded into $L_\infty$. A Lipschitz extension of $f$ to a mapping $\tilde{f}: L_\infty \rightarrow \ell^N_2$ should act like the identity $\ell^N_2$, so the techniques of [8] should yield a linear projection from $L_\infty$ onto $\ell^N_2$ whose norm is of order $\|f\|_{lip}$. Since $\ell^N_2$ is complemented in $L_\infty$ only of order $\sqrt{N}$ and there are $\epsilon$-nets for $S^{N-1}$ of cardinality $n = \lfloor 4/\epsilon \rfloor^N$, we should get that

$$L(\infty,n) \geq \sqrt{N} \geq \delta \left( \frac{\log n}{\log \epsilon} \right)^{1/2}.$$

In Theorem 2 we make this approach work when $\epsilon$ is of order $N^{-2}$, so we get

$$L(\infty,n) \geq \delta^t \left( \frac{\log n}{\log \log n} \right)^{1/2}.$$

That the difficulties we incur with the outlined approach for larger values of $\epsilon$ are not purely technical is the gist of the following extension result.

(*)THEOREM 2. Suppose that $X$ is a metric space, $A \subset X$, $f:A \rightarrow \ell_2$ is Lipschitz and $d(x,y) \geq \epsilon > 0$ for all $x \neq y \in A$. Then there is an extension $\tilde{f}: X \rightarrow \ell_2$ of $f$ so that

$$\|\tilde{f}\|_{lip} \leq \frac{6D}{\epsilon} \|f\|_{lip},$$

where $D$ is the diameter of $A$.

PROOF. We can assume by translating $f$ that there is a point $0 \in A$ so that $f(0) = 0$. Set $B = A - \{0\}$ and define

$$F : A \rightarrow \ell^1_B$$

by

$$F(b) = \begin{cases} \delta_{\epsilon, b}, & b \neq 0 \\ 0, & b = 0 \end{cases}.$$

Define

$$G : \ell^1_B \rightarrow \ell_2$$

by

$$G( \sum_{b \in B} \alpha_{\epsilon, b} \delta_b ) = \sum_{b \in B} \alpha_{\epsilon, b} f(b).$$

(*) See the appendix for a generalization of Theorem 2 proved by Yoav Benyamini.
Then
\[ G F = f, \ G \text{ is linear with} \]
\[ \|G\| \leq D \|f\|_{\ell^1}, \text{ and } \|F\|_{\ell^1} \leq 2/\varepsilon. \]

A weakened form of Grothendieck's inequality (see section 2.6 in [9])
yields that \( G \) (as any bounded linear operator from an \( L_1 \) space into a
Hilbert space) factors through an \( \ell_\infty(N) \) space:
\[ G = H J, \ |J| = 1, \|H\| \leq 3 \|G\|, \]
\[ J : \ell^B_1 \to \ell_\infty(N), \ H : \ell_\infty(N) \to \ell_2. \]

By the non-linear Hahn-Banach Theorem the mapping \( J F \) has an extension
\[ E : X \to \ell_\infty(N) \text{ which satisfies} \]
\[ \|E\|_{\ell^1} \leq \|J F\|_{\ell^1} \leq 2/\varepsilon. \]

Then \( \tilde{E} = H E \) extends \( f \) and \( \|\tilde{E}\|_{\ell^1} \leq 6D \varepsilon \|f\|_{\ell^1} \), as desired. \( \Box \)

For the proof of Theorem 3, we need three well known facts which we state
as lemmas.

**Lemma 2.** Suppose that \( Y, X \) are normed spaces and \( f : S(Y) \to X \) is Lipschitz
with \( f(0) = 0 \). Then the positively homogeneous extension of \( f \), defined for
\( y \in Y \) by
\[ \tilde{f}(y) = \|y\| f \left( \frac{y}{\|y\|} \right), \ (y \neq 0); \quad \tilde{f}(0) = 0 \]
is Lipschitz and
\[ \|\tilde{f}\|_{\ell^1} \leq 2 \|f\|_{\ell^1} + \|f\|_{\ell^\infty}. \]

**Proof.** Given \( y_1, y_2 \in Y \) with \( 0 < \|y_1\| \leq \|y_2\|, \)
\[ \|\tilde{f}(y_1) - \tilde{f}(y_2)\| \leq \|y_1\| f \left( \frac{y_1}{\|y_1\|} \right) - \|y_2\| f \left( \frac{y_2}{\|y_2\|} \right) + \|y_2\| \| f \left( \frac{y_1}{\|y_1\|} \right) - f \left( \frac{y_2}{\|y_2\|} \right) \|
\]
\[ = \left( \|y_2\| - \|y_1\| \right) \| f \left( \frac{y_1}{\|y_1\|} \right) \| + \|y_2\| \|f\|_{\ell^1} \| \left( \frac{y_1}{\|y_1\|} - \frac{y_2}{\|y_2\|} \right) \|
\]
\[ \leq \|y_1 - y_2\| \|f\|_{\ell^\infty} + \|f\|_{\ell^1} \| \left( \frac{y_2}{\|y_1\|} - y_2 \right) \|
LEMMA 3. If $Y$ is an $n$-dimensional Banach space and $0 < \varepsilon$, then $S(Y)$ admits an $\varepsilon$-net of cardinality at most $(1 + 4/\varepsilon)^n$.

PROOF. Let $M$ be a subset of $S(Y)$ maximal with respect to $\|x - y\| \geq \varepsilon$ for all $x \neq y \in M$.

Then the sets

$$b(y, \varepsilon/2) \cap S(Y), \quad (y \in M)$$

are pairwise disjoint hence so are the sets

$$b(y, \varepsilon/4), \quad (y \in M).$$

Since these last sets are all contained in $b(1 + \varepsilon/4)$, we have that

$$\text{card } M \cdot \text{vol } b(\varepsilon/4) \leq \text{vol } b(1 + \varepsilon/4)$$

so that

$$\text{card } M \leq \left[\frac{4}{\varepsilon} (1 + \varepsilon/4)\right]^n. \quad \square$$

LEMMA 4. There is a constant $\delta > 0$ so that for each $1 \leq p < 2$ and each $N = 1, 2, \ldots$, $L^p$ contains a subspace $E$ such that

$$d(E, \ell^N_2) \leq 2$$

and every projection from $L^p$ onto $E$ has norm at least

$$\delta N \frac{1}{1/p - 1/2}.$$

PROOF. Given a finite dimensional Banach space $X$ and $1 \leq p < \infty$, let

$$\gamma_p(X) = \inf \{\|T\| : T : X \to L^p, \; S : L_p \to X, \; S \circ T = I_X\}.$$

So $\gamma_p(X)$ is the projection constant of $X$, hence by [4], [12]

$$\gamma_1(\ell^N_2) = \gamma_p(\ell^N_2) = \sqrt{2n/\pi}.$$

This gives the $p = 1$ case.
For $1 < p < 2$ we reduce to the case $p = 1$ by using Example 3.1 of [2],
which asserts that there is a constant $C < \infty$ so that for $1 \leq p < 2$ $\ell_p^N$ contains a subspace $E$ with $d(E, \ell_2^N) \leq 2$. Since, obviously,
\[
d(\ell_p^N, \ell_2^N) \leq (CN)^{1 - 1/p}
\]
we get that if $E$ is $K$-complemented in $\ell_p^N$, then
\[
\pi^{-1/2} (2n)^{1/2} = \gamma_1(\ell_2^N) \leq d(E, \ell_2^N) d(\ell_p^N, \ell_2^N) K
\]
\[
\leq 2 (CN)^{1 - 1/p} K. \quad \square
\]

The next piece of background information we need for Theorem 3 is a linearization result which is an easy consequence of the results in [8].

**Proposition 1.** Suppose $X \subset Y$ and $Z$ are Banach spaces, $f : Y \to Z$ is Lipschitz, and $U : X \to Z$ is bounded, linear. Then there is a linear operator $G : Z^* \to Y^*$ so that $\|G\| \leq \|f\|_{\ell_1^p}$ and
\[
\|R_2 G - U^*\| \leq \|f_X - U\|_{\ell_1^p},
\]
where $R_2$ is the natural restriction map from $Y^*$ onto $X^*$.

**Remark.** Note that if $Z$ is reflexive, the mapping $F = G^*|_Y : Y \to Z$ satisfies $\|F\| \leq \|f\|_{\ell_1^p}$ and $\|F |_X - U\| \leq \|f |_X - U\|_{\ell_1^p}$.

**Proof.** We first recall some notation from [8]. If $Y$ is a Banach space, $Y^\theta$ denotes the Banach space of all scalar valued Lipschitz functions $y^\theta$ from $Y$ for which $y^\theta(0) = 0$, with the norm $\|y^\theta\|_{\ell_1^p}$. There is an obvious isometric inclusion from $Y^*$ into $Y^\theta$. For a Lipschitz mapping $f : Y \to Z$, $Z$ a normed space, we can define a linear mapping
\[
f^\theta : Z^* \to Y^\theta \text{ by } f^\theta z = z f.
\]
Given Banach spaces $X \subset Y$, Theorem 2 of [8] asserts that there are norm one linear projections
\[
P_Y : Y^\theta \to Y^*, \quad P_X : X^\theta \to X^*
\]
so that
\[
P_X R_1 = R_2 P_Y,
\]
where $R_1$ is the restriction mapping from $Y^*$ onto $X^*$. Thus if $X \subset Y$, $f$, $U$, $Z$ are as in the hypothesis of Proposition 1, the linear mapping $P_Y f^*$ satisfies

$$
\|P_Y f^*\| \leq \|f\|_{\text{lip}}, \quad R_Z P_Y f^* = P_X R_1 f^*.
$$

Since $U: X \to Z$ is linear,

$$
U^* = P_X U^*
$$

so

$$
\|R_Z P_Y f^* - U^*\| = \|P_X (R_1 f^* - U^*)\|
$$

$$
\leq \|R_1 f^* - U^*\| = \sup_{z^* \in S(Z^*)} \|R_1 f^* z^* - U^* z^*\|
$$

$$
= \sup_{z^* \in S(Z^*)} \| (z^* f) \| \leq \|f\|_{X^*} \|U\|_{\text{lip}}. \quad \square
$$

The final lemma we use in the proof of Theorem 3 is a smoothing result for homogeneous Lipschitz functions.

**Lemma 5.** Suppose $X \subset Y$ and $Z$ are Banach spaces with $\dim X = k < \infty$, $F: Y \to Z$ is Lipschitz with $F$ positively homogeneous (i.e. $F(\lambda y) = \lambda F(y)$ for $\lambda \geq 0$, $y \in Y$) and $U: X \to Z$ is linear. Then there is a positively homogeneous Lipschitz mapping

$$
\mathcal{F}: Y \to Z
$$

which satisfies

1. $\|\mathcal{F} \|_{X^*} \leq (6k + 2) \|F\|_{S(X^*)} \|U\|_{S(X^*)}$

2. $\|\mathcal{F}\|_{\text{lip}} \leq 4 \|F\|_{\text{lip}}$.

**Proof.** For $y \in S(Y)$ define

$$
\mathcal{F} y = \int_{B_X(1)} F(y + x) \, d\mu(x)
$$

where $\mu(\cdot)$ is Haar measure on $X (= \mathbb{R}^k)$ normalized so that

$$
\mu(B_X(1)) = 1.
$$

For $y_1, y_2 \in S(Y)$ we have
\[ \| \hat{F} y_1 - \hat{F} y_2 \| \leq \int_{B_X(1)} \| F(y_1 + x) - F(y_2 + x) \| \, d\mu(x) \]
\[ \leq \| F \|_{\ell^p} \| y_1 - y_2 \| \]

so

\[ \| \hat{F} \|_{\ell^p} \leq \| F \|_{\ell^p} . \]

For \( x_1, x_2 \in S(X) \) with \( \| x_1 - x_2 \| = \delta > 0 \) we have, since \( U \) is linear, that

\[ \| (\hat{F} - U) x_1 - (\hat{F} - U) x_2 \| = \]

\[ \| \int_{B_X(1)} F(x_1 + x) \, d\mu(x) - \int_{B_X(1)} U(x_1 + x) \, d\mu(x) - \int_{B_X(1)} F(x_2 + x) \, d\mu(x) + \]

\[ \int_{B_X(1)} U(x_2 + x) \, d\mu(x) \| \leq \]

\[ \leq \int_{B_X(1)} \| Fx - Ux \| \, d\mu(x) \]

\[ \leq \sup_{x \in B_X(2)} \| Fx - Ux \| \mu [B_X(x_1; 1) \Delta B_X(x_2; 1)] \]

\[ = 2 \sup_{x \in B_X(1)} \| Fx - Ux \| \mu [B_X(x_1; 1) \Delta B_X(x_2; 1)] \quad \text{[since } F \text{ is positively homogeneous]} \]

Since

\[ B_X(x_1; 1) \Delta B_X(x_2; 1) \subset [B_X(x_1; 1) \sim B_X(x_1; 1-5)] \cup [B_X(x_2; 1 \sim B_X(x_2; 1-5)] \]

we have if \( \delta \leq 1 \) that

\[ \mu [B_X(x_2; 1) \Delta B_X(x_2; 1)] \leq 2 [1 - (1-5)^{k}] \]

\[ \leq 2 \, k \, \delta \]

and hence for all \( x_1, x_2 \in S(X) \) that

\[ \| (\hat{F} - U) x_1 - (\hat{F} - U) x_2 \| \leq 4k \| F \|_{S(X)} - U \|_{S(X)} \| x_1 - x_2 \| \]

whence
Finally, note that the positive homogeniety of $F$ implies that

$$\|F\|_\infty \leq 2 \|F\|_{L^p}$$

and

$$\|\hat{F}\|_\infty \leq 2 \|\hat{F}\|_{L^p} \leq 2 \|F\|_\infty \leq 2 \|F\|_{L^p}.$$

It now follows from Lemma 2 that the positively homogeneous extension $\hat{F}$ of $F$ satisfies the conclusions of Lemma 5.

**THEOREM 3.** There is a constant $\tau > 0$ so that for all $n = 2, 3, 4, \ldots$ and all $1 \leq p < 2$,

$$L(p, n) \geq \tau \left( \frac{\log n}{\log \log n} \right)^{1/p - 1/2}.$$

**REMARK.** Since $L(\infty, n) \geq L(1, n)$, we get the lower estimate for $L(\infty, n)$ mentioned in the introduction.

**PROOF.** Given $p$ and $n$, for a certain value of $N = N(n)$ to be specified later choose a subspace $E$ of $L_p$ with $d(E, \ell_2^N) \leq 2$ and $E$ only $\delta N^{1/p - 1/2}$-complemented in $L_p$ (Lemma 4). For a value $\varepsilon = \varepsilon(n) > 0$ to be specified later, let $A$ be a minimal $\varepsilon$-net of $S(E)$, so, by Lemma 3,

$$\text{card } A \leq (1 + 4/\varepsilon)^N.$$

One relation among $n$, $N$, $\varepsilon$ we need is

$$1 \leq n \leq (1 + 4/\varepsilon)^N.$$

Let $f : A \cup \{0\} \to E$ be the identify map. Since $d(E, \ell_2^N) \leq 2$, we can by Lemma 2 get a positively homogeneous extension $\hat{f} : L_p \to E$ of $f$ so that

$$\|\hat{f}\|_{L^p} \leq 6 L(p, n).$$

Since $\hat{f}(a) = f(a) = a$ for $a \in A$ and $A$ is an $\varepsilon$-net for $S(E)$, we get that for $x \in S(E)$,

$$\|\hat{f}(x) - x\| \leq (6 L(p, n) + 1) \varepsilon.$$

Therefore, from Lemma 5 we get a Lipschitz mapping $\hat{f} : L_p \to E$ which satisfies

$$\|\hat{f}\|_{L^p} \leq 24 L(p, n).$$

(1.5)

$$\|\hat{f}\|_{L^p} \leq (8N + 2)(6 L(p, n) + 1) \varepsilon.$$
Note that if

\[(1.6) \quad (8N + 2)(6L(p,n) + 1)\epsilon \leq 1/2,\]

\[(1.5) \quad \text{implies that there is a linear projection from } L_p \text{ onto } E \text{ with norm at most } 48L(p,n), \text{ so we can conclude that} \]

\[L(p,n) > \frac{5}{48}N^{1/p} - 1/2.\]

Finally, we just need to observe that (1.4) and (1.6) are satisfied (at least for sufficiently large \(n\)) if we set

\[\epsilon = \frac{\log n}{2 \log \log n}, \quad N = \frac{\log n}{2 \log \log n}.
\]

2. OPEN PROBLEMS.

Besides the obvious question left open by the preceding discussion (i.e. whether the estimate for \(L(\omega, n)\) given in Theorem 1 is indeed the best possible), there are several other problems which arise naturally in the present context. We mention here only some of them.

PROBLEM 1. \text{Is it true that for } 1 < p < 2, \text{ every subset } X \text{ of } L(0,1), \text{ and every Lipschitz map } f \text{ from } X \text{ into } \ell_2^k \text{ there is an extension } \tilde{f} \text{ of } f \text{ from } L_p(0,1) \text{ into } \ell_2^k \text{ with}

\[(2.1) \quad \|\tilde{f}\|_{\ell_1^p} \leq C(p)\|f\|_{\ell_1^p} k^{1/p} - 1/2\]

where \(C(p)\) depends only on \(p\)?

A positive answer to problem 1 combined with Lemma 1 above will of course provide an alternative proof to the result of Marcus and Pisier [10] mentioned in the introduction. The linear version of problem 1 (where \(X\) is a subspace and \(f\) a linear operator) is known to be true (cf. [7] and [3]).

PROBLEM 2. \text{What happens in the Marcus-Pisier theorem if } 2 < p < \infty? \text{ Is the Lipschitz analogue of Maurey's extension theorem [11] (cf. also [3]) true? In other words, is it true that for } 2 < p < \infty \text{ there is a } c(p) \text{ such that for every Lipschitz map } f \text{ from a subset } X \text{ of } L_p(0,1) \text{ into } \ell_2 \text{ there is a Lipschitz extension } \tilde{f} \text{ from } L_p(0,1) \text{ into } \ell_2 \text{ with}

\[\|\tilde{f}\|_{\ell_1^p} \leq c(p)\|f\|_{\ell_1^p} ?\]
PROBLEM 3. What are the analogues of Lemma 1 in the setting of Banach spaces different from Hilbert spaces? The most interesting special case seems to be concerning the spaces $\ell^n_m$. It is well known that every finite metric space $X = \{x_i\}_{i=1}^n$ embeds isometrically into $\ell^n_m$ (the point $x_i$ is mapped to the $n$-tuple $(d(x_1, x_i), d(x_2, x_i), \ldots, d(x_n, x_i))$ in $\ell^n_m$). Hence in view of Lemma 1 it is quite natural to ask the following. Does there exist for all $\varepsilon > 0$ (or alternatively for some $\varepsilon > 0$) a constant $K(\varepsilon)$ so that for every metric space $X$ with cardinality $n$ there is a Banach space $Y$ with $\dim Y \leq K(\varepsilon) \log n$ and a map $f$ from $X$ into $Y$ so that

$$
\|f\|_{\text{lip}} \|f^{-1}\|_{\text{lip}} \leq 1 + \varepsilon.
$$

A weaker version of Problem 3 is

PROBLEM 4. It is true that for every metric space $X$ with cardinality $n$ there is a subset $\tilde{X}$ in $\ell^2$ and a Lipschitz map $F$ from $X$ onto $\tilde{X}$ so that

$$
\|F\|_{\text{lip}} \|F^{-1}\|_{\text{lip}} \leq K \sqrt{\log n}
$$

for some absolute constant $K$?

Since for every Banach space $Y$ with $\dim Y = k$ we have $d(Y, \ell^k_2) \leq \sqrt{k}$ (cf. [6]) it is clear that a positive answer to problem 3 implies a positive answer to problem 4. V. Milman pointed out to us that it follows easily from an inequality of Enflo (cf. [1]) that (2.2), if true, gives the best possible estimate. (In the notation of [1], observe that the "m-cube"

$$
x_\theta = (\theta_1, \theta_2, \ldots, \theta_m) (\theta \in \{-1, 1\}^m)
$$

in $\ell^m_1$ has all "diagonals" of length $2m$ and all "edges" of length 2, so that if $F$ is any Lipschitz mapping from these $2^m$ points in $\ell^m_1$ into a Hilbert space, the corollary in [1] implies that

$$
\|F\|_{\ell^m_1} \|F^{-1}\|_{\ell^m_1} \geq m^{1/2}.
$$

3. APPENDIX.

After this note was written, Yoav Benyamini discovered that Theorem 2 remains valid if $\ell^2_2$ is replaced with any Banach space. He kindly allowed us to reproduce here his proof. The main lemma Benyamini uses is:

LEMA 6. Let $\Gamma$ be an indexing set and let $\{e_\gamma\}_{\gamma \in \Gamma}$ be the unit vector basis for $c_0(\Gamma)$. Set
Then

(i) there is a retraction $G$ from $\ell_\infty(\Gamma)$ onto $B$ which satisfies

$\|G\|_{\text{lip}} \leq 2$

(ii) there is a mapping $H$ from $\ell_\infty(\Gamma)$ into $A$ which satisfies

$\|H\|_{\text{lip}} \leq 4$ and $He = e_\gamma$ for all $\gamma \in \Gamma$.

PROOF. Since the mapping $x \mapsto x^+$ is a contractive retraction from $\ell_\infty(\Gamma)$ onto its positive cone, $\ell_\infty(\Gamma)^+$, to prove (i) it is enough to define $G$ only on $\ell_\infty(\Gamma)^+$.

For $y \in \ell_\infty(\Gamma)^+$, let

$g(y) = \inf \{ t : \| (y - te)^+ \|_1 \leq 1 \}$

where $e \in \ell_\infty(\Gamma)$ is the function identically equal to one and $\| \cdot \|_1$ is the usual norm in $\ell_1(\Gamma)$. Clearly the inf is actually a minimum and $0 \leq g(y) \leq \| y \|_\infty$. Note that

$|g(y) - g(z)| \leq \| y - z \|_\infty$.

Indeed, assume that $g(y) \geq g(z)$. Then

$y - [g(z) + \| y - z \|_\infty e] \leq y - g(z)e + z - y \leq z - g(z)e$

and hence

$\| (y - [g(z) + \| y - z \|_\infty e]^e) \|_1 \leq 1$;

that is

$g(y) \leq g(z) + \| y - z \|_\infty$.

Now set for $y \in \ell_\infty(\Gamma)^+$

$G(y) = (y - g(y)e)^+$.

To prove (ii), it is enough, in view of (i), to define $H$ on $B$ with $\|H\|_{B} \|_{\text{lip}} \leq 2$. For $y \in B$, $y = \{ y(\gamma) \}_{\gamma \in \Gamma}$, defined $H_y$ by

$H_y(\gamma) = (2y(\gamma) - 1)^+$.
For \( y \in B \), there is at most one \( \gamma \in \Gamma \) for which \( y(\gamma) > \frac{1}{2} \), hence \( HB \subset A \). Evidently \( H_\gamma = e_\gamma \) for \( \gamma \in \Gamma \) and \( \|H\|_{\ell^1} \leq 2 \).

**THEOREM 2** (Y. Benyamini). Suppose that \( X \) is a metric space, \( Y \) is a subset of \( X \) with \( d(x,y) \geq \varepsilon > 0 \) for all \( x \neq y \in Y \), \( Z \) is a Banach space, and \( f: Y \to Z \) is Lipschitz. Then there is an extension \( \tilde{f}: X \to Z \) of \( f \) so that

\[
\|\tilde{f}\|_{\ell^1} \leq (4D/\varepsilon)\|f\|_{\ell^1}
\]

where \( D \) is the diameter of \( Y \).

**PROOF.** Represent

\( Y = \{0\} \cup \{y_\gamma : \gamma \in \Gamma\} \)

and assume, by translating \( f \), that \( f(0) = 0 \). We can factor \( f \) through the subset \( C = \{0\} \cup \{e_\gamma : \gamma \in \Gamma\} \) of \( \ell^\infty(\Gamma) \) by defining \( g: Y \to C \), \( h: C \to Z \) by

\[
g(y_\gamma) = e_\gamma, \quad g(0) = 0
\]

\[
h(e_\gamma) = f(y_\gamma), \quad h(0) = 0.
\]

Evidently,

\[
\|g\|_{\ell^1} \leq 1/\varepsilon, \quad \|h\|_{\ell^1} \leq D\|f\|_{\ell^1}.
\]

By the non-linear Hahn-Banach theorem, \( g \) has an extension to a function \( \tilde{g}: X \to \ell^\infty(\Gamma) \) with \( \|\tilde{g}\|_{\ell^1} = \|g\|_{\ell^1} \), so to complete the proof, it suffices to extend \( h \) to a function \( \tilde{h}: B \to Z \) with \( \|\tilde{h}\|_{\ell^1} = \|h\|_{\ell^1} \) and apply Lemma 6(ii).

Define for \( 0 \leq t \leq 1 \) and \( \gamma \in \Gamma \)

\[
\tilde{h}(te_\gamma) = th(e_\gamma).
\]

If \( 1 \geq t \geq s \geq 0 \) and \( \gamma \neq \Delta \in \Gamma \) then

\[
\|\tilde{h}(te_\gamma) - \tilde{h}(se_\Delta)\| \leq (t-s)\|h(e_\gamma)\| + s \|h(e_\Delta) - h(e_\gamma)\|
\]

\[
\leq (t-s)\|h\|_{\ell^1} + s\|h\|_{\ell^1} = \|h\|_{\ell^1}\|te_\gamma - se_\Delta\|_{\ell^\infty},
\]

so \( \|\tilde{h}\|_{\ell^1} = \|h\|_{\ell^1} \).
REFERENCES


William B. Johnson
The Ohio State University and
Texas A & M University

Joram Lindenstrauss
The Hebrew University of Jerusalem,
Texas A & M University, and
The Ohio State University