# CS205b/CME306

Lecture 5

## 1 Modeling 1D Materials

## 2 Springs

### 2.1 Simple Spring

One of the simplest forces one might imagine is the simple spring. A simple spring may be formulated in 1D with one endpoint fixed to the origin and the other of mass $m$ located at $x$. Let $x_0$ be the rest length of the spring and $k_s$ and $k_d$ be spring-specific constants. The force applied by the spring is

$$F = -k_s\left(\frac{x}{x_0} - 1\right) - k_d v$$

The constant $k_s$ is the spring constant, which has units of newtons $N = kgms^{-2}$. In 1D, this is just Young's modulus, but in higher dimensions the two differ by a geometry term. The constant $k_d$ is the damping coefficient and has units of $kgs^{-1}$. Note that $x/x_0 - 1$ is the strain on the spring and $F$ is its stress, so that the spring equation (ignoring the damping term) expresses a stress-strain relationship.

We can write the spring as a first order system as

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{mx_0} & -\frac{k_d}{m} \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k_s}{m} \end{pmatrix}.$$

The eigenvalues of this system are

$$\lambda = -\frac{k_d}{2m} \pm \sqrt{\left(\frac{k_d}{2m}\right)^2 - \frac{k_s}{mx_0}}$$

We can simplify analysis of the eigenvalues by rewriting them in terms of a dimensionless $k_{d_0}$ as

$$k_d = k_{d_0}\sqrt{\frac{mk_s}{x_0}} \qquad \lambda = \frac{-k_{d_0} \pm \sqrt{k_{d_0}^2 - 4}}{2}\sqrt{\frac{k_s}{mx_0}}.$$

The system is under-damped if $k_{d_0} < 2$. If $k_{d_0} = 0$ the system has no damping at all, and its eigenvalues will be pure imaginary. The system is critically damped and has a repeated eigenvalues when $k_{d_0} = 2$, and the system is over-damped when $k_{d_0} > 2$. As before, $|\lambda|\Delta t \approx 1$.

## 2.2 Elastic Materials

By connecting masses and springs, it is possible to simulate elastic materials. In 1D, an elastic material is modeled by interconnecting a line of masses with springs. In 2D, an elastic material is modeled by splitting it into triangles with masses at the vertices and springs along the edges. In 3D, the same can be done using tetrahedra with masses at the vertices and springs along the edges. In the limit of infinite stiffness, these materials will behave as rigid bodies, and in the limit of no stiffness, the particles move without interacting.

Many materials may be modeled quite effectively using only masses and springs. There are other effective ways to model materials, such as the finite element method, which instead computes forces between particles by considering the deformation and material properties of the volumetric elements. Finite elements tend to require fewer elements, but they are also slower. The finite element method has better determined elastic behavior and permits arbitrary constitutive models, including fracture and plasticity. It is also more accurate, though accuracy has little meaning when considering collisions and fracture. We will cover finite elements more later.