CS205b/CME306

Lecture 2

1 Conservation of Mass

Conservation of mass may be derived by examining the density of a control volume Ω with boundary $\partial\Omega$ as in Figure 1. The density at any point in space is ρ, and it moves with velocity u. The mass of this control volume is

$$\text{mass} = \int_{\Omega} \rho \, dV.$$

Any change in this mass is due to material leaving and entering the control volume. Let n be the outward-facing (unit) normal of a small patch of the boundary $\partial\Omega$. Then, $n \cdot u$ is the rate of movement across the boundary at any point (with positive indicating movement out of the volume). The remaining velocity component $u - (n \cdot u)n$ is due to flow along the boundary but not across it. If the surface patch has area A, then the rate of mass flow across the boundary patch is $A(n \cdot u)\rho = (\rho u) \cdot dS$, where dS is the surface element. The total of this flow across the boundary is the rate of mass decrease, since a positive value indicates flow out of the boundary. This leads to the statement of mass conservation

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \, dV = - \int_{\partial\Omega} (\rho u) \cdot \, dS.$$

This is called the weak form, since it does not involve spatial derivatives. Strong form is obtained by applying the divergence theorem to the right hand side.

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \, dV = - \int_{\Omega} \nabla \cdot (\rho u) \, dV.$$

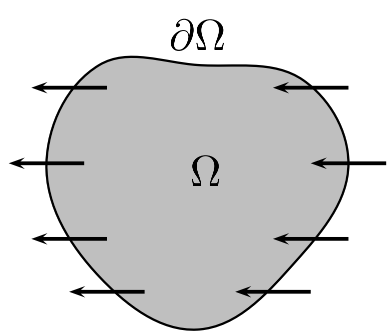

Figure 1: Control volume Ω with boundary $\partial\Omega$ through which material flows.

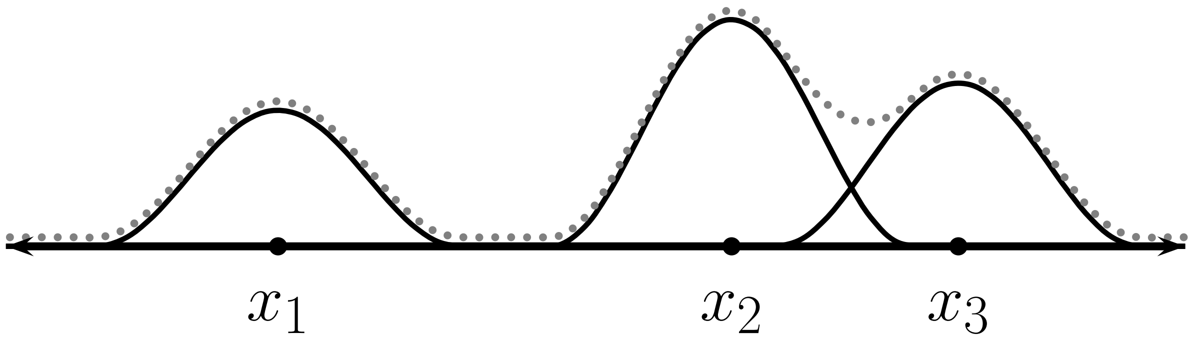

Figure 2: Three chunks located at x_1, x_2, and x_3. The black curves show the density contribution of a single chunk at each point in space. The dotted line shows the density profile of space.

By moving the partial time derivative into the integral, the two integrals may be merged

$$\int_\Omega \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \, dV = 0.$$

Finally, this must be true of any control volume Ω, so that the strong form is obtained

$$\rho_t + \nabla \cdot (\rho u) = 0.$$

If we restrict ourselves to 1D, we can derive this result in another way. In 1D, the control volume is an interval $\Omega = [a, b]$. The rate of increase in material across the endpoint a is $\rho_R u_R$, and the rate of increase across the endpoint b is $\rho_R u_R$, since positive velocity represents material leaving the control volume. This expresses conservation of mass as

$$\frac{\partial}{\partial t} \int_\Omega \rho \, dV = \frac{\partial}{\partial t} \int_{[a,b]} \rho \, dx = -(\rho_R u_R - \rho_L u_L).$$

Defining interval width $\Delta x = b - a$ and the average density $\overline{\rho} = \frac{1}{\Delta x} \int_{[a,b]} \rho \, dx$, this becomes

$$\frac{\partial \overline{\rho}}{\partial t} = -\frac{\rho_R u_R - \rho_L u_L}{\Delta x}.$$

In the limit of $a \to b$, we have $\Delta x \to 0$. If we let $\rho = \rho_R$ and $u = u_R$, then $\overline{\rho} \to \rho$. The limit of the right hand side is the classical definition of a derivative. Putting these together yields

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x}.$$

2 Smoothed Particle Hydrodynamics (SPH)

If we consider mass as attached to chunks, we automatically conserve mass as we move the chunks around. However, because the laws governing motion often involve derivatives of quantities stored on the chunks, it is useful to have a definition of these properties everywhere in space that is sufficiently differentiable. This can be achieved by spreading attributes associated with chunks, such as their mass, over some local region of space. This is the basic idea behind the SPH method.

Let $W(x)$ be some sufficiently differentiable function that can be used to spread attributes over space. In particular, consider a chunk centered at x_i with mass m_i. Then define the density contribution from this chunk at any point x in space to be $\rho(x) = m_i W(x - x_i)$. Because we would like to conserve mass, we insist that the total mass spread throughout space is the mass assigned to that chunk

$$m_i = \int_{-\infty}^{\infty} \rho(x)\, dx = \int_{-\infty}^{\infty} m_i W(x - x_i)\, dx$$

which leads to the requirement

$$\int_{-\infty}^{\infty} W(x)\, dx = 1.$$

It is also desirable for $W(x)$ to be symmetric about the origin (so that density is spread evenly in all directions about its center of mass). It is also more efficient if $W(x)$ has local influence in that $W(x) = 0$ everywhere outside some region around the origin. Note that $W(x)$ has units of one over volume, since it yields unity when integrated over the volume of space.