CS205b/CME306

Lecture 17

1 Incompressible Flow

1.1 MAC Grid

Supplementary Reading: Osher and Fedkiw, §18.1, §18.2

Recall that the system of equations we must solve for incompressible flow is

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0 \tag{2}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g}. \tag{3}$$

$$\tag{4}$$

Harlow and Welch [2] proposed the use of a special grid for incompressible flow computations. This specially defined grid decomposes the computational domain into cells with velocities defined on the cell faces and scalars defined at cell centers. That is, in 2D, $p_{i,j}$, $\rho_{i,j}$ are defined at cell centers while $u_{i\pm\frac{1}{2},j}$ and $v_{i,j\pm\frac{1}{2}}$ are defined at the appropriate cell faces.

Equation (2) is solved by first defining the cell center velocities with simple averaging

$$u_{i,j} = \frac{u_{i-\frac{1}{2},j} + u_{i+\frac{1}{2},j}}{2}$$

$$v_{i,j} = \frac{v_{i,j-\frac{1}{2}} + v_{i,j+\frac{1}{2}}}{2}$$

Then the spatial derivatives are evaluated in a straightforward manner, for example using 3rd order accurate Hamilton-Jacobi ENO. The temporal derivative can be evaluated with a TVD RK scheme.

In order to update the velocity based on equation (3), we first need u and v at all the cell faces. Again, we obtain the values by simple averaging. For example,

$$v_{i-\frac{1}{2},j} = \frac{1}{4}\Big(v_{i-1,j-\frac{1}{2}} + v_{i-1,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} + v_{i,j+\frac{1}{2}}\Big).$$

Similarly, to get u values on the v faces, we compute the average

$$u_{i,j-\frac{1}{2}} = \frac{1}{4}\Big(u_{i-\frac{1}{2},j-1} + u_{i+\frac{1}{2},j-1} + u_{i-\frac{1}{2},j} + u_{i+\frac{1}{2},j}\Big).$$

The term MAC Grid stands for Marker-And-Cell. It refers to what the discretization was first used for rather than describing the discretization itself.

1.2 Discretizing Divergence of Velocity

Supplementary Reading: Osher and Fedkiw, §18.3, §23.1

Here we discuss the discretization in 8 of the term $\nabla \cdot \mathbf{u}^\star$. Since we are solving for the pressure, which on the MAC grid lives at the cell centers, we need to discretize the term at the cell centers. We have that

$$\begin{aligned}(\nabla \cdot \mathbf{u}^\star)_{i,j} &= \left(u^\star_x + v^\star_y\right)_{i,j} \\ &= \frac{u^\star_{i+\frac{1}{2},j} - u^\star_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^\star_{i,j+\frac{1}{2}} - v^\star_{i,j-\frac{1}{2}}}{\Delta y} + O(\Delta x^2) + O(\Delta y^2)\end{aligned}$$

So we have used the intermediate velocity stored at the cell faces to get a second order accurate approximation to $\nabla \cdot \mathbf{u}^\star$ at the cell centers.

1.3 Semi-Lagrangian Velocity Advection

Evolving the momentum equation 3 is done by first advecting velocities and applying body forces with 6 to obtain $\mathbf{u}^\star$. Then, equation 6 is evaluated by solving the elliptic Poisson's equation, followed by applying the resulting pressure to obtain a final velocity $\mathbf{u}^{n+1}$. There is no time step restriction for second step, so the only CFL restriction is on the velocity advection. Therefore, if we use the semi-Lagrangian method for the velocity advection we can eliminate the remaining time step restriction. For $u^\star$, the method is

$$u^\star_j = u^n\left(\mathbf{x}_j - \mathbf{u}^n_j \Delta t\right)$$

For $v^\star$ we must also account for gravity, so we have

$$v^\star_j = v^n(\mathbf{x}_j - \mathbf{u}^n_j \Delta t) - \Delta t\, g$$

where we are computing

$$v_t + \mathbf{u} \cdot \nabla v = -g.$$

This is a Godunov splitting, which is first order accurate.

1.4 Projection Method

In order to update the velocity, we use the projection method due to Chorin [1]. The projection method is applied by first computing an intermediate velocity field $\mathbf{u}^\star$ ignoring the pressure term,

$$\frac{\mathbf{u}^\star - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n = \mathbf{g}^n, \tag{5}$$

and then computing a divergence free velocity field $\mathbf{u}^{n+1}$,

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^\star}{\Delta t} + \frac{\nabla p^{n+1}}{\rho^{n+1}} = 0, \tag{6}$$

using the pressure as a correction. Note that combining equations 5 and 6 to eliminate $\mathbf{u}^\star$ results in equation 3.

Taking the divergence of equation 6 results in

$$\nabla \cdot \left(\frac{\nabla p^{n+1}}{\rho^{n+1}} \right) = \frac{\nabla \cdot \mathbf{u}^\star}{\Delta t} \tag{7}$$

after setting $\nabla \cdot \mathbf{u}^{n+1} = 0$, i.e. after assuming that the new velocity field is divergence free. equation 7 defines the pressure in terms of the value of Δt used in equation 5. Defining a scaled pressure of $p^\star = p\Delta t$ leads to

$$\mathbf{u}^{n+1} - \mathbf{u}^\star + \frac{\nabla p^\star}{\rho^{n+1}} = 0$$

and

$$\nabla \cdot \left(\frac{\nabla p^\star}{\rho^{n+1}} \right) = \nabla \cdot \mathbf{u}^\star$$

in place of equations 6 and 7 where $p^\star$ does not depend on Δt. When the density is spatially constant, we can define $\hat{p} = p\Delta t/\rho$ leading to

$$\mathbf{u}^{n+1} - \mathbf{u}^\star + \nabla \hat{p} = 0$$

and

$$\Delta \hat{p} = \nabla \cdot \mathbf{u}^\star \tag{8}$$

where $\hat{p}$ does not depend on Δt or ρ.

This method utilizes the Helmholtz-Hodge decomposition of the vector field $\mathbf{u}^\star$,

$$\mathbf{u}^\star = \mathbf{u}^{n+1} + \nabla \hat{p}.$$

In general, the Helmholtz-Hodge decomposition of a vector field expresses the vector field as a divergence free vector field plus the gradient of a scalar field.

1.5 Computing Boundary Conditions

We have shown how to discretize the Poisson equation and handle Dirichlet and Neumann boundary conditions. We have not yet said much about how to obtain these boundary conditions. A typical scenerio that leads to a Dirichlet boundary condition is a free surface, such as the surface of the water in a glass. In this case, the pressure will be 1 atm, the pressure that the air applies to the surface of the water. The other type of boundary condition that occurs in the example of a glass of water is the boundary condition between the water and the walls of the container. This boundary condition may be described by requiring the velocity component at the cell face of the boundary to be equal to the velocity of the wall itself. If the walls of the container are stationary, then this velocity component will be zero. Unfortunately, this is a condition on velocity, not a condition on pressure as is required by the Poisson discretization. To obtain this boundary condition, we begin with 6. Lets also assume that the boundary condition is on the side of the container, so that we are considering the x direction, so that we should consider

$$\frac{u^{n+1} - u^\star}{\Delta t} + \frac{p_x}{\rho} = 0.$$

For this face we have $u^{n+1} = u_{BC}$, which simply states that we should be computing a velocity that agrees with the velocity u_{BC} of the container.

$$\frac{u_{BC} - u^\star}{\Delta t} + \frac{p_x}{\rho} = 0.$$

Solving this for the pressure derivative yields

$$p_x = -\rho \frac{u_{BC} - u^\star}{\Delta t}.$$

This procedure may be simplified somewhat by observing that we may simply enforce the boundary conditions on $u^\star$, so that it need not be computed but can just be assigned $u^\star = u_{BC}$. Then, the corresponding pressure condition becomes $p_x = 0$.

1.6 Algorithm Overview

We now have the following steps in updating the velocity field for incompressible flow using the projection method:

1. Compute the intermediate velocity field $\mathbf{u}^\star$ (at cell faces)

$$\frac{\mathbf{u}^\star - \mathbf{u}^n}{\Delta t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} \tag{9}$$

2. Solve an elliptic equation for the pressure (at cell centers)

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \frac{\nabla \cdot \mathbf{u}^\star}{\Delta t} \tag{10}$$

3. Compute the divergence free velocity field $\mathbf{u}^{n+1}$ (at cell faces)

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^\star}{\Delta t} + \frac{\nabla p}{\rho} = 0 \tag{11}$$

We can multiply p by Δt, to get $p^\star = p\Delta t$, and rewrite steps 2 and 3 as

2a.

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p^\star \right) = \nabla \cdot \mathbf{u}^\star \tag{12}$$

3a.

$$\mathbf{u}^{n+1} - \mathbf{u}^\star + \frac{\nabla p^\star}{\rho} = 0 \tag{13}$$

If ρ is constant, then we can move it under the ∇ operator and defining $\hat{p} = \frac{p\Delta t}{\rho}$, we can rewrite steps 2 and 3 as

2b.

$$\Delta \hat{p} = \nabla \cdot \mathbf{u}^\star \tag{14}$$

3b.

$$\mathbf{u}^{n+1} - \mathbf{u}^\star + \nabla \hat{p} = 0 \tag{15}$$

References

[1] Chorin, A., *A Numerical Method for Solving Incompressible Viscous Flow Problems*, J. Comput. Phys. 2, 12-26 (1967).

[2] Harlow, F. and Welch, J., *Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with a Free Surface*, The Physics of Fluids 8, 2182-2189 (1965).