CS205b/CME306

Lecture 15

1 Incompressible Flow

Supplementary Reading: Osher and Fedkiw, §18.1, §18.2

Recall the stability condition for compressible flow

$$\max_{\Omega}\{|u + c|, |u|, |u - c|\} < \frac{\Delta x}{\Delta t}$$

where the quantity on the left of the inequality is the physical wave speed and the quantity on the right is the numerical wave speed. Then the time step is given by

$$\Delta t = \alpha \frac{\Delta x}{\max_{\Omega}\{|u + c|, |u|, |u - c|\}}$$

where α is the CFL number, $\alpha < 1$.

For example, we might have $u = 1, c = 300$, so that

$$|u + c| = 301, \qquad |u| = 1, \qquad |u - c| = 299.$$

Observe that the $u \pm c$ fields impose a much more severe restriction on the time step than the u field. If $|u| \ll |c|$ and we only care about the linear flow phenomena, i.e., the phenomena corresponding to the u field, then we can avoid this difficulty by modeling the flow as incompressible. The assumption of incompressibility is valid in the limit as $\frac{c}{u} \to \infty$ and is equivalent to the divergence free condition $\nabla \cdot \mathbf{u} = 0$. In fact, the definition of incompressibility for a velocity field $\mathbf{u}$ is that $\nabla \cdot \mathbf{u} = 0$.

Modeling the flow as incompressible allows us to eliminate the severe time step restriction due to the $u \pm c$ fields, and focus on the u field. As a result, we lose the nonlinear behavior (e.g., shocks, rarefactions) associated with the $u \pm c$ fields.

1.1 Equations

Starting from conservation of mass, momentum and energy, the equations for incompressible flow are derived using the divergence free condition, $\nabla \cdot \mathbf{u} = 0$, which implies that there is no compression or expansion in the flow field. Note that in 1D, this is just $u_x = 0$, which implies that u is spatially constant.

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1.1.1 Conservation of Mass

In 1D, the equation for conservation of mass is

$$\rho_t + (\rho u)_x = 0.$$

Applying the chain rule, we get

$$\rho_t + \rho_x u + \rho u_x = 0.$$

Since the flow is incompressible, $\nabla \cdot \mathbf{u} = 0$ which reduces to $u_x = 0$ in 1D, so that the equation is simply

$$\rho_t + u\rho_x = 0.$$

In multiple dimension, the equation is given by

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0.$$

1.1.2 Conservation of Momentum

Starting with the equation for conservation of mass,

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

we then apply the chain rule to get

$$\rho_t u + \rho u_t + \rho u u_x + u(\rho u)_x + p_x = 0.$$

We combine the first and fourth terms

$$u(\rho_t + (\rho u)_x) + \rho u_t + \rho u u_x + p_x = 0.$$

Note that the quantity in parentheses is 0 from conservation of mass, so that

$$\rho u_t + \rho u u_x + p_x = 0.$$

By incompressibility, the second term is 0, so that we are left with

$$\rho u_t + p_x = 0.$$

Dividing by ρ, we get

$$u_t + \frac{p_x}{\rho} = 0.$$

In multiple dimension, the equation is given by

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = 0.$$

1.1.3 Conservation of Energy

The equation for conservation of energy in 1D is

$$E_t + [(E + p)u]_x = 0.$$

Substituting $E = \rho e + \frac{1}{2}\rho u^2$, we get

$$\left(\rho e + \frac{1}{2}\rho u^2\right)_t + \left[\left(\rho e + \frac{1}{2}\rho u^2 + p\right)u\right]_x = 0.$$

Differentiating, we have

$$\left(e + \frac{1}{2}u^2\right)\rho_t + \rho e_t + \rho u u_t + \left(\rho e + \frac{1}{2}\rho u^2 + p\right)u_x + \left(e + \frac{1}{2}u^2\right)u\rho_x + \rho u e_x + \rho u u_x + u p_x = 0$$

Since $u_x = 0$, this becomes

$$\left(e + \frac{1}{2}u^2\right)\rho_t + \rho e_t + \rho u u_t + \left(e + \frac{1}{2}u^2\right)u\rho_x + \rho u e_x + u p_x = 0$$

Rearranging terms, we have

$$\left(e + \frac{1}{2}u^2\right)(\rho_t + u\rho_x) + u\rho\left(u_t + \frac{p_x}{\rho}\right) + \rho e_t + \rho u e_x = 0$$

By the equations for conservation of mass and conservation of momentum, this reduces to

$$\rho e_t + \rho u e_x = 0$$

Dividing by ρ, we get

$$e_t + u e_x = 0$$

In multiple dimensions the equation for conservation of energy is

$$e_t + \mathbf{u} \cdot \nabla e = 0$$

In summary, the equations for incompressible flow (in multiple spatial dimensions) are

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = 0$$
$$e_t + \mathbf{u} \cdot \nabla e = 0$$

1.1.4 Decoupling from Energy

Recall that for compressible flow, we had an equation of state $p = p(\rho, e)$. If p does not happen to depend on the internal energy e, then neither neither the equation for conservation of mass nor the equation for conservation of momentum will depend on e. This means it is possible to compute ρu and ρ as a coupled system, then compute e offline if the energy is desired. We shall see later that

in the case of incompressible flow, the pressure does not depend on the internal energy. We do, however, need to retain $\nabla \cdot \mathbf{u} = 0$ as an equation that we must solve. We are left with the system

$$\nabla \cdot \mathbf{u} = 0$$
$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0 \tag{1}$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = 0 \tag{2}$$
$$\tag{3}$$

Body forces, e.g. gravity, are added to the RHS of the momentum equation, so that it becomes

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g}$$

where $\mathbf{g} = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$.

In the case of incompressible flow, it is often the case that the density of the initial configuration may be assumed constant, such as is often the case for air or water. If this is the case, then $\nabla \rho = 0$, so that $\rho_t + \mathbf{u} \cdot \nabla \rho = 0$ reduces to $\rho_t = 0$ and implies that the density remains constant throughout the simulation. In this way, the density equation may often also be ignored.

1.1.5   Equation of State

Consider the case where the initial density ρ and internal energy e are spatially constant in the initial conditions. Then, $\nabla \rho = 0$ and $\nabla e = 0$, so that the laws for conservation of mass and energy imply $\rho_t = 0$ and $e_t = 0$. Thus, both density and internal energy are constant in space and time. Since $p = p(\rho, e)$ depends only on the density and internal energy, it too must be constant in space and time. The only occurance of the pressure is in the equation for conservation of mass, $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = 0$, which only depends on $\nabla p = 0$. This leaves us to solve $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0$. Unfortunately, solving this will generally not lead us to satisfy $\nabla \cdot \mathbf{u} = 0$, so we are stuck. Incompressible flow does not have an equation of state in the way compressible flow did.

The problem is that $\nabla \cdot \mathbf{u} = 0$ has a global influence that travels through the domain instantly. While the system for compressible flow was hyperbolic, the system for incompressible flow is instead elliptic. Rather than information traveling through the domain at finite speeds, we instead find that information travels through the domain with infinite speed. The solution to the problem lies in the two remaining equations, the divergence free condition and conservation of momentum. If we take the divergence of the conservation of mass equation we get

$$
\begin{aligned}
0 &= \nabla \cdot (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho}) \\
&= (\nabla \cdot \mathbf{u})_t + \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \cdot \left(\frac{\nabla p}{\rho} \right) \\
&= \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \cdot \left(\frac{\nabla p}{\rho} \right) \\
\nabla \cdot \left(\frac{\nabla p}{\rho} \right) &= -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})
\end{aligned}
$$

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This is an equation similar to the Poisson equation, which we may solve for the pressure p. This equation replaces the equation of state for computing the pressure and ensures that we maintain the constraint $\nabla \cdot \mathbf{u} = 0$. As suggested earlier, this does not depend on the internal energy e, so the internal energy may be ignored if it is not otherwise needed. If it is the case that ρ is spatially constant, it may be pulled from inside the divergence operator and moved to the other side of the equation to yield the Poisson equation

$$\Delta p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}).$$