CS205b/CME306

Lecture 13

1 Multiple Spatial Dimensions

In multiple spatial dimensions, the ENO discretization is applied independently using a dimension by dimension discretization. For example, consider a two dimensional conservation law

$$\phi_t + f(\phi)_x + g(\phi)_y = 0$$

on a rectangular 2-D grid. Here, we sweep through the grid from bottom to top performing ENO on 1-D horizontal rows of grid points to evaluate the $f(\phi)_x$ term. The $g(\phi)_y$ term is evaluated in a similar manner sweeping through the grid from left to right performing ENO on 1-D vertical rows of grid points. Once we have a numerical approximation to each of the spatial terms, we update the entire equation in time with a method of lines approach using, for example, a TVD Runge-Kutta method.

We emphasize that dimension by dimension discretization is *not* the same as dimensional splitting, such as the first order Godunov splitting and second order Strang splitting. In dimension by dimension discretization, the fluxes in each dimension are evaluated independently, but the time stepping is still coupled.

2 Systems of Conservation Laws

Supplementary Reading: Osher and Fedkiw, §14.5.1, §14.5.2

In general, a hyperbolic system will simultaneously contain a mixture of processes: smooth bulk convection and wave motion, and discontinuous processes involving contacts, shocks and rarefactions. For example, if a gas in a tube is initially prepared with a jump in the states (density, velocity and temperature) across some surface, as the evolution proceeds in time these jumps will break up into a combination of shocks, rarefactions and contacts, in addition to any bulk motion and sound waves that may exist or develop. This is called a shock tube experiment.

The hyperbolic systems we encounter in physical problems are written in what are effectively the mixed variables where the apparent behavior is quite complicated. A transformation is required to decouple them back into unmixed fields that exhibit the pure contact, shock and rarefaction phenomena (as well as bulk convection and waves). In a real system, this perfect decoupling is not possible because the mixing is nonlinear, but it can be linearized over a small space and time region, and this provides the basis for the theoretical understanding of the structure of general hyperbolic systems of conservation laws. This is called a transformation to characteristic variables. As we shall see, this transformation also provides the basis for designing appropriate numerical methods.

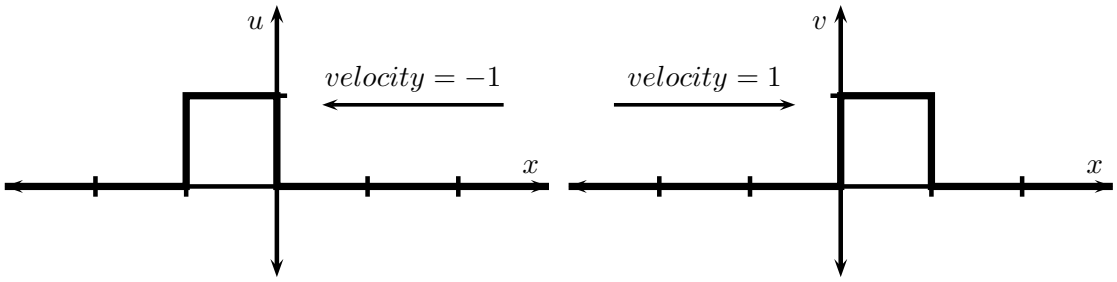

Figure 1: The solution is the initial data for u moving to the left with speed 1, and the initial data for v moving to the right with speed 1.

Consider a simple hyperbolic system of N equations

$$\boldsymbol{\phi}_t + [\mathbf{f}(\boldsymbol{\phi})]_x = 0 \tag{1}$$

in one spatial dimension. The basic idea of characteristic numerical schemes is to transform this nonlinear system to a system of N (nearly) independent scalar equations of the form

$$\phi_t + \lambda \phi_x = 0$$

and discretize each scalar equation independently in an upwind biased fashion using the characteristic velocity λ. Then transform the discretized system back into the original variables.

2.1 Example

We start with an example of two separate scalar equations and show how we can change variables to write them as a coupled system. Consider the two equations

$$\begin{cases} u_t - u_x = 0 \\ v_t + v_x = 0 \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases}$$

The analytic solution is

$$u(x,t) = u_0(x+t)$$
$$v(x,t) = v_0(x-t)$$

For example, Figure 1 depicts the solution for the initial data given below.

$$u_0(x) = \begin{cases} 1, & x \in (-1,0) \\ 0, & \text{otherwise} \end{cases}$$
$$v_0(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

Next we make the change of variables

$$w = v + u$$
$$z = v - u$$

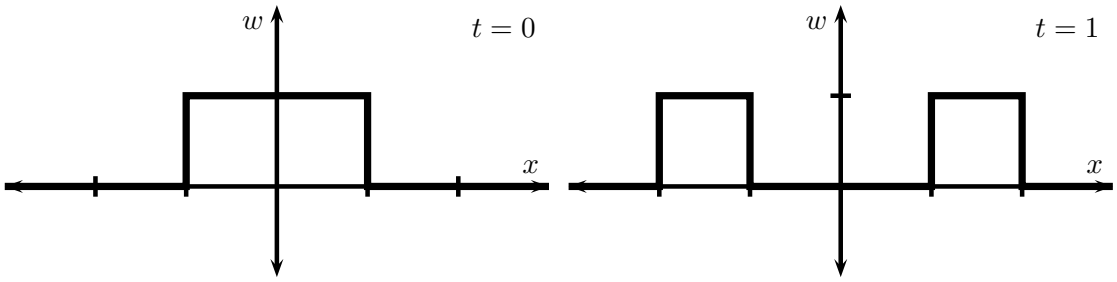

Figure 2: The solution consists of two separate components, one moving to the left, and the other moving to the right.

This gives

$$w_t = v_t + u_t = -v_x + u_x = -z_x$$
$$z_t = v_t - u_t = -v_x - u_x = -w_x$$

So u and v are independent of each other, but w and z depend on each other. The system for w and z can be written as

$$\begin{pmatrix} w \\ z \end{pmatrix}_t + \begin{pmatrix} z \\ w \end{pmatrix}_x = 0.$$

The solution is given by

$$w(x,t) = v_0(x-t) + u_0(x+t)$$
$$z(x,t) = v_0(x-t) - u_0(x+t)$$

The graph for w is shown in Figure 2.

This demonstrates that although the picture for w may appear complicated, the underlying solutions u and v are simply two waves moving to the left and right.

Now we rewrite the system as

$$\begin{pmatrix} w \\ z \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}_x = 0$$

which is in the form

$$\boldsymbol{\phi}_t + \mathbf{J}\boldsymbol{\phi}_x = 0.$$

Similarly, we can write the system (1) in quasilinear form as

$$\boldsymbol{\phi}_t + \mathbf{f}'(\boldsymbol{\phi})\boldsymbol{\phi}_x = 0.$$

Here $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\phi}}$. Recall that in the scalar case

$$\phi_t + f(\phi)_x = 0$$

where we had the quasilinear form

$$\phi_t + f'(\phi)\phi_x = 0$$

the characteristic speed was given by $f'(\phi)$. For the case of systems, the characteristic speeds are given by the eigenvalues of the Jacobian, $\mathbf{J}$.

Coming back to our example, we have

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We compute the eigenvalues:

$$\det(\mathbf{J} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

So the eigenvalues of $\mathbf{J}$ are

$$\lambda_1 = -1, \lambda_2 = 1.$$

Next we determine the eigenvectors. For $\lambda_1 = -1$, we have $\mathbf{J}\mathbf{R}_1 = \lambda_1 \mathbf{R}_1$. We solve for $\mathbf{R}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} b \\ a \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

Hence $\mathbf{R}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution. For $\lambda_2 = 1$, we have $\mathbf{J}\mathbf{R}_2 = \lambda_2 \mathbf{R}_2$. We solve for $\mathbf{R}_2 = \begin{pmatrix} c \\ d \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

Hence $\mathbf{R}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution. Therefore, we have computed that

$$\mathbf{J}\begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Note that if $\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \end{pmatrix}$ is a matrix whose columns are the right eigenvectors of $\mathbf{J}$, and $\mathbf{L} = \mathbf{R}^{-1}$ is taken to be the matrix whose rows are left eigenvectors of $\mathbf{J}$,

$$\mathbf{J}\mathbf{R} = \mathbf{R}\Lambda \qquad \mathbf{L}\mathbf{J} = \Lambda\mathbf{L} \qquad \mathbf{L}\mathbf{R} = \mathbf{R}\mathbf{L} = \mathbf{I}.$$

In summary, we have computed the eigensystem for our example, and we can use this to transform $\mathbf{J}$ into diagonal form,

$$\mathbf{L}\mathbf{J}\mathbf{R} = \Lambda.$$

It is important to note that if a system is hyperbolic, $\mathbf{J}$ will have N real eigenvalues λ_p, $p = 1, \ldots, N$, and N linearly independent right eigenvectors. Once the eigensystem is determined, we can use it to diagonalize the matrix $\mathbf{J}$.

Suppose we want to discretize our equation at the node x_0, where $\mathbf{L}$ and $\mathbf{R}$ have values $\mathbf{L}_0$ and $\mathbf{R}_0$. To get a locally diagonalized form, we multiply our system equation by the constant matrix $\mathbf{L}_0$ which nearly diagonalizes $\mathbf{J}$ over the region near x_0. We require a constant matrix so that we can move it inside all derivatives to obtain

$$[\mathbf{L}_0\boldsymbol{\phi}]_t + \mathbf{L}_0\mathbf{J}\mathbf{R}_0[\mathbf{L}_0\boldsymbol{\phi}]_x = 0$$

where we have inserted $\mathbf{I} = \mathbf{R}_0\mathbf{L}_0$ to put the equation in a more recognizable form. The spatially varying matrix $\mathbf{L}_0\mathbf{J}\mathbf{R}_0$ is exactly diagonalized at the point x_0, with eigenvalues $\lambda_{0,p}$, and it is nearly diagonalized at nearby points. Thus the equations are sufficiently decoupled for us to apply upwind biased discretizations independently to each component with $\lambda_{0,p}$ determining the upwind biased direction for the p-th component equation. Once this system is fully discretized, we multiply the entire system by $\mathbf{L}_0^{-1} = \mathbf{R}_0$ to return to the original variables.

In terms of our original equation 1, our procedure for discretizing at a point x_0 is simply to multiply the entire system by the left eigenvector matrix $\mathbf{L}_0$,

$$[\mathbf{L}_0\boldsymbol{\phi}]_t + [\mathbf{L}_0\mathbf{f}(\boldsymbol{\phi})]_x = 0$$

and discretize the $p = 1, \ldots, N$ scalar components of this system

$$[(\mathbf{L}_0\boldsymbol{\phi})_p]_t + [(\mathbf{L}_0\mathbf{f}(\boldsymbol{\phi}))_p]_x = 0$$

independently, using upwind biased differencing with the upwind direction for the p-th equation determined by the sign of λ_p. We then multiply the resulting spatially discretized system of equations by $\mathbf{R}_0$ to recover the spatially discretized fluxes for the original variables

$$\boldsymbol{\phi}_t + \mathbf{R}_0\Delta(\mathbf{L}_0\mathbf{f}(\boldsymbol{\phi})) = 0$$

where Δ stands for the upwind biased discretization operator, i.e. either the ENO-RF or ENO-LLF discretization.

We call λ_p the p-th characteristic velocity or speed, $(\mathbf{L}_0\boldsymbol{\phi})_p = \mathbf{L}_{0,p} \cdot \boldsymbol{\phi}$ the p-th characteristic state or field (here $\mathbf{L}_p$ denotes the p-th row of $\mathbf{L}$, i.e. the p-th left eigenvector of $\mathbf{J}$), and $(\mathbf{L}_0\mathbf{f}(\boldsymbol{\phi}))_p = \mathbf{L}_{0,p} \cdot \mathbf{f}(\boldsymbol{\phi})$ the p-th characteristic flux. According to the local linearization, it is approximately true that the p-th characteristic field rigidly translates in space at the p-th characteristic velocity. Thus this decomposition corresponds to the local physical propagation of independent waves or signals.

2.2 Discretization

Assume that the system (1) has N equations. Then the Jacobian, $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\phi}}$, will be and $N \times N$ matrix. Furthermore, we know that $\mathbf{J}$ is diagonalizable, since the system is hyperbolic. Let us denote the eigenvalues, right eigenvectors, and left eigenvectors of $\mathbf{J}$ as λ^p, $\mathbf{R}^p$, $\mathbf{L}^p$ respectively for $p = 1, \ldots, N$. Recall from the previous lecture that we can choose the left and right eigenvectors so that for

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}^1 & \mathbf{R}^2 & \cdots & \mathbf{R}^N \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} \mathbf{L}^1 \\ \mathbf{L}^2 \\ \vdots \\ \mathbf{L}^N \end{pmatrix}$$

the relation $RL = LR = \mathbf{I}$ holds. That is, $\mathbf{R}$ and $\mathbf{L}$ are chosen to be inverses.

In the previous lecture we looked at a linear, constant coefficient system. In that case, the Jacobian was a constant matrix. In general, the Jacobian, and hence its eigensystem, will be spatially varying.

As in the scalar case, our discretization is of the form

$$(\boldsymbol{\phi}_i)_t + \frac{\mathbf{f}_{i+1/2} - \mathbf{f}_{i-1/2}}{\Delta t} = 0.$$

For grid point i, we need to compute the numerical flux functions at $x_{i+1/2}$ and $x_{i-1/2}$. Let us look in detail at computing $F_{i+1/2}$.

The first step is to evaluate the eigensystem at the point $x_{i+1/2}$. Since we only have $\boldsymbol{\phi}$ at the grid points, we obtain $\boldsymbol{\phi}$ at the cell walls using the standard average, $\boldsymbol{\phi}_{i+1/2} = (\boldsymbol{\phi}_i + \boldsymbol{\phi}_{i+1})/2$. Then for p = 1, ..., N, we find the component of the numerical flux function in the p-th characteristic field. In the p-th characteristic field we have an eigenvalue $\lambda^p(\boldsymbol{\phi}_{i+1/2})$, left eigenvector $\mathbf{L}^p(\boldsymbol{\phi}_{i+1/2})$, and right eigenvector $\mathbf{R}^p(\boldsymbol{\phi}_{i+1/2})$. We put $\boldsymbol{\phi}$ values and $\mathbf{f}(\boldsymbol{\phi})$ values into the p-th characteristic field by taking the dot product with the left eigenvector,

$$u_i = \mathbf{L}^p(\boldsymbol{\phi}_{i+1/2}) \cdot \boldsymbol{\phi}_i$$

$$f_i = \mathbf{L}^p(\boldsymbol{\phi}_{i+1/2}) \cdot \mathbf{f}(\boldsymbol{\phi}_i)$$

where u_i and f_i are scalars. Once in the characteristic field we perform a scalar version of the conservative ENO scheme obtaining a scalar numerical flux function $\mathscr{F}^p_{i+1/2}$ in the scalar field. We take this flux out of the characteristic field by multiplying with the right eigenvector,

$$\mathscr{F}^p_{i+1/2} = \mathscr{F}^p_{i+1/2} \mathbf{R}^p(\boldsymbol{\phi}_{i+1/2})$$

where $\mathscr{F}^p_{i+1/2}$ is the portion of the numerical flux function $\mathscr{F}_{i+1/2}$ from the p-th field. Once we have evaluated the contribution to the numerical flux function from each field, we get the total numerical flux by summing the contributions from each field,

$$\mathscr{F}_{i+1/2} = \sum_p \mathscr{F}^p_{i+1/2}$$

completing the evaluation of our numerical flux function at the point $x_{i+1/2}$.