CS205b/CME306

Lecture 12

1 Discrete Conservation Form

1.1 ENO-Roe Discretization

The ENO-Roe discretization uses the weak form of a hyperbolic conservation law. This distinguishes it from the Hamilton Jacobi ENO discretization described earlier, which was based on the strong form. This makes ENO-Roe suitable for simulating nonlinear phenomena such as shocks, where the solutions are discontinuous. The ENO-Roe discretization is based on the discrete conservation form and is computed by using a divided difference table to compute the numerical fluxes as described earlier. We present here a more detailed description of the computation of these flux functions in 1D.

For a specific cell wall, located at $x_{i_0+1/2}$, we find the associated numerical flux function $\mathscr{F}_{i_0+1/2}$ as follows. First, we define a *characteristic speed*

$$\lambda_{i_0+1/2} = f'(\phi_{i_0+1/2}).$$

1.1.1 Burgers' Equation

For example, recall Burgers' equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

The flux is given by

$$f(u) = \frac{u^2}{2} \qquad f'(u) = f_u(u) = u.$$

Therefore,

$$\lambda(x) = f'(u(x)) = u(x).$$

The value of u at the half grid points is defined using a standard linear average

$$u_{i_0+1/2} = (u_{i_0} + u_{i_0+1})/2.$$

1.1.2 Interpolation and Computing Numerical Flux

If $\lambda_{i_0+1/2} > 0$, set $k = i_0$. Otherwise, set $k = i_0 + 1$. Next, define

$$Q_1(x) = (D_k^1 H)(x - x_{i_0+1/2}).$$

If $|D^2_{k-1/2}H| \le |D^2_{k+1/2}H|$, then $c = D^2_{k-1/2}H$ and $k^\star = k-1$. Otherwise, $c = D^2_{k+1/2}H$ and $k^\star = k$. Define

$$Q_2(x) = c(x - x_{k-1/2})(x - x_{k+1/2}).$$

If $|D^3_{k^\star}H| \le |D^3_{k^\star+1}H|$, then $c^\star = D^3_{k^\star}H$. Otherwise, $c^\star = D^3_{k^\star+1}H$. Define

$$Q_3(x) = c^\star(x - x_{k^\star-1/2})(x - x_{k^\star+1/2})(x - x_{k^\star+3/2}).$$

Then

$$\mathscr{F}_{i_0+1/2} = H'(x_{i_0+1/2}) = Q_1'(x_{i_0+1/2}) + Q_2'(x_{i_0+1/2}) + Q_3'(x_{i_0+1/2})$$

which simplifies to

$$\mathscr{F}_{i_0+1/2} = D^1_k H + c(2(i_0-k)+1)\Delta x + c^\star(3(i_0-k^\star)^2 - 1)\Delta x^2.$$

1.2 ENO-RF Discretization (ENO-Roe with the Entropy Fix)

The ENO-Roe discretization can admit entropy violating expansion shocks near sonic points. That is, at a place where a characteristic velocity changes sign (a sonic point) it is possible to have a stationary expansion shock solution with a discontinuous jump in value. If this jump were smoothed out even slightly, it would break up into an expansion fan (i.e. rarefaction) and dissipate, which is the desired physical solution. For a specific cell wall, $x_{i_0+1/2}$, if there are no nearby sonic points, then we use the ENO-Roe discretization. Otherwise, we add high order dissipation to our calculation of $\mathscr{F}_{i_0+1/2}$ to break up any entropy violating expansion shocks. We call this entropy fixed version of the ENO-Roe discretization ENO-Roe Fix (ENO-RF). More specifically, we use $\lambda_{i_0} = f'(\phi_{i_0})$ and $\lambda_{i_0+1} = f'(\phi_{i_0+1})$ to decide if there are sonic points in the vicinity. If λ_{i_0} and λ_{i_0+1} agree in sign, we use the ENO-Roe discretization where $\lambda_{i_0+1/2}$ is taken to be the same sign as λ_{i_0} and λ_{i_0+1}. Otherwise we use the ENO-local Lax Friedrichs (ENO-LLF) entropy fix discretization given below. ENO-LLF is applied at both expansions where $\lambda_{i_0} < 0$ and $\lambda_{i_0+1} > 0$ and at shocks where $\lambda_{i_0} > 0$ and $\lambda_{i_0+1} < 0$. While this adds extra numerical dissipation at shocks, it is not harmful as shocks are self-sharpening. In fact, this extra dissipation provides some viscous regularization which is especially desirable in multiple spatial dimensions. For this reason, authors sometimes use the ENO-LLF method everywhere as opposed to mixing in ENO-Roe discretizations where the upwind direction is well determined by the eigenvalues λ.

1.3 ENO-LLF Discretization

The ENO-LLF discretization is formulated as follows. Consider two primitive functions H^+ and H^-. We compute a divided difference table for each of them with their first divided differences being

$$D^1_i H^\pm = f(\phi_i) \pm \alpha_{i_0+1/2}\phi_i \qquad \alpha_{i_0+1/2} = \max(|\lambda_{i_0}|, |\lambda_{i_0+1}|)$$

is our dissipation coefficient, and controls the amount of dissipation added. Note that the dissipation coefficient, $\alpha_{i_0+1/2}$, is determined locally for each cell wall, hence the name ENO-local Lax Friedrichs. (One could also construct a scheme where a global dissipation coefficient α is used, a global Lax Friedrichs, but this generally adds too much dissipation).

The second and third divided differences, $D^2_{i+1/2}H^\pm$ and $D^3_i H^\pm$ are then defined in the standard way, like those of H.

For H^+, set $k = i_0$. Then, replacing H with H^+ everywhere, define $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and finally $\mathscr{F}^+_{i_0+1/2}$ using the ENO-Roe algorithm above. For H^-, set $k = i_0 + 1$. Then, replacing H with H^- everywhere, define $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and finally $\mathscr{F}^-_{i_0+1/2}$ again by using the ENO-Roe algorithm above. Finally,

$$\mathscr{F}_{i_0+1/2} = \frac{\mathscr{F}^+_{i_0+1/2} + \mathscr{F}^-_{i_0+1/2}}{2}$$

is the new numerical flux function with added high order dissipation.