# CS205b/CME306

Lecture 10

## 1 Hyperbolic Conservation Laws

**Supplementary Reading:** Osher and Fedkiw, Section §14.1.1, §14.1.2, §14.1.3, §14.1.4; Leveque §11.6, §12.9, §12.10, §12.11

A physical system is described by the laws of conservation of mass, momentum and energy. The integral form of the conservation law is derived by considering a fixed control volume. Let us denote the control volume by $\Omega$, and its boundary by $\partial\Omega$. If $\phi$ represents the conserved quantity, then the total amount of $\phi$ in the control volume is given by

$$\int_\Omega \phi \, dV$$

The rate of change of $\phi$ in the control volume is given by the flux through the region boundary, plus whatever internal sources exist.

$$\frac{d}{dt}\int_\Omega \phi \, dV = -\int_{\partial\Omega} \mathbf{f}(\phi) \cdot dS + \int_\Omega s(\phi) \, dV$$

The flux can be either convective or diffusive. The distinction is that diffusive fluxes are driven by gradients, while convective fluxes persist even in the absence of gradients. As an example of a diffusive flux, consider the opening of a perfume bottle. The gradient in concentration of the perfume causes it to diffuse. For most flows where compressibility is important, e.g. flows with shock waves, one needs only to model the convective transport and can ignore diffusion (mass diffusion, viscosity and thermal conductivity) as well as the source terms (such as chemical reactions, atomic excitations, and ionization processes). Moreover, convective transport requires specialized numerical treatment while diffusive and reactive effects can be treated with standard numerical methods—such as simple central differencing—that are independent of those for the convective terms. A source term might include creation of the quantity through a chemical reaction. Conservation laws with only convective fluxes are known as *hyperbolic conservation laws*.

The *weak form* of the conservation law is usually written as

$$\frac{d}{dt}\int_\Omega \phi \, dV + \int_{\partial\Omega} \mathbf{f}(\phi) \cdot dS = \int_\Omega s(\phi) \, dV$$

The equation now resembles the linear advection equation we looked at previously.

We now consider the *strong form* (or differential form) of the conservation law. The strong form can be derived from the weak form by taking an infinitesimally small control volume and applying the divergence theorem. The equation is then written as

$$\phi_t + \nabla \cdot \mathbf{f}(\phi) = s(\phi)$$

The strong form may not always hold, as it requires that $\nabla \cdot \mathbf{f}(\phi)$ exist. The strong form is not valid when there is a shock, contact discontinuity, or when the function is not smooth. These are the types of phenomena we would like to consider.

The first thing to realize is that the presence of discontinuities poses a limitation on the order of accuracy of any numerical scheme we might devise. There is a conjecture that states that we cannot get a scheme with higher than first order accuracy. However, we are still interested in higher order methods such as ENO, because even if our scheme is limited to first order accuracy overall, in many parts of the domain the dominant error term will be the higher order one. For example, we may have an error that looks like $C_1\Delta x + C_2\Delta x^3$, with $C_1 \ll C_2$ almost everywhere in our domain, so that the higher order term dominates for the time step size taken. This is called a *high resolution* method.

The important physical phenomena exhibited by hyperbolic conservation laws are:

1. bulk convection and waves
2. contact discontinuities
3. shocks
4. rarefactions

We will briefly describe the physical features and mathematical model equations for each effect, and most importantly note the implications they have on the design of numerical methods. The first two phenomena are linearly degenerate because they can be modeled locally by the linear advection equation. The last two phenomena are genuinely nonlinear.

## 1.1 Bulk Convection and Waves

Bulk convection is simply the bulk movement of matter from one spot to another, like water streaming from a hose. Waves are small amplitude smooth disturbances that transmit through the system without any bulk transport, like ripples on a water surface or sound waves through air. Whereas convective transport occurs at the gross velocity of the material, waves propagate at the speed of sound in the system (relative to the bulk convective motion of the system). Waves interact by superposition, so that they can either cancel out (interfere) or enhance each other.

The simplest model equation that describes bulk convective transport is the linear convection equation

$$\phi_t + \mathbf{u} \cdot \nabla\phi = 0 \tag{1}$$

where $\mathbf{u}$ is a constant equal to the convection velocity. The solution to this is simply that $\phi$ translates at the constant speed $\mathbf{u}$. This same equation can also be taken as a simple model of wave motion, if $\phi$ is a sine wave and $\mathbf{u}$ is interpreted as the speed of sound. The linear convection equation is also an important model for understanding smooth transport in any conservation law. As long as $\mathbf{f}$ is smooth and $\phi$ has no jumps in it, the general scalar conservation law

$$\phi_t + \nabla \cdot \mathbf{f}(\phi) = 0 \tag{2}$$

can be rewritten as

$$\phi_t + \mathbf{f}'(\phi) \cdot \nabla\phi = 0 \tag{3}$$

where $\mathbf{f}'(\phi)$ acts as a convective velocity. That is, locally in smooth parts of the flow, a conservation law behaves like bulk convection with velocity $\mathbf{f}'(\phi)$. This is called the local characteristic velocity of the flow. For systems, the term $\mathbf{f}'(\phi)$ is the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$.

Note, however, that one must be extremely careful going from equation 2 to equation 3. In doing so, we are assuming that $f$ depends on $x$ through $\phi$ only. For example, consider the equation for conservation of mass in one dimension.

$$\rho_t + (\rho u)_x = 0.$$

The chain rule gives

$$\rho_t + u\rho_x + \rho u_x = 0.$$

However, applying (3) with $f(\rho) = \rho u$ would give

$$\rho_t + u\rho_x = 0$$

which is a linearization. It assumes that $u_x = 0$, or that $f$ depends on $x$ through $\rho$ only, which will not be true in general. However, this formulation can be used to gain intuition and as a guide for devising numerical schemes.

### 1.2 Contact Discontinuities

A contact discontinuity is a persistent, discontinuous jump in mass density moving by bulk convection through the system. Since there is negligible mass diffusion, such a jump persists. These jumps usually appear at the point of contact of different materials, for example, a contact discontinuity separates oil from water. Contacts move at the local bulk convection speed, or more generally the characteristic speed, and can be modeled by using step-function initial data in equation 1. Since contacts are simply a bulk convection effect, they retain any perturbations they receive. Thus we expect contacts to be especially sensitive to numerical methods, i.e. any spurious alteration of the contact will tend to persist and accumulate.

Note that there is no discontinuity in pressure or velocity across the contact discontinuity, but only in density.

### 1.3 Shock Waves

A shock is a spatial jump in material properties, like pressure and temperature, that develops spontaneously from smooth distributions and then persists. The shock jump is self-forming and also self-maintaining. This is unlike a contact discontinuity which must be put in the system initially and will not re-sharpen itself if it is smeared out by some other process. Shocks develop through a feedback mechanism in which strong impulses move faster than weak ones, and thus tend to steepen themselves up into a step profile as they travel through the system. Familiar examples are the "sonic boom" of a jet aircraft, or the "bang" from a gun. These sounds are our perceptions of a sudden jump in air pressure.

the simplest model equation that describes shock formation is the one dimensional Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

which looks like the convection equation with a non-constant convective speed of $u$, i.e. $u_t + uu_x = 0$. Note that this equation is nonlinear since the velocity depends on the conserved variable $u$ and has the weak form

$$\int_{[x_l,x_r]} u_t \, dx + \int_{\partial[x_l,x_r]} \frac{u^2}{2} \, dx = \int_{[x_l,x_r]} u_t \, dx + \frac{u_r^2}{2} - \frac{u_l^2}{2} = 0.$$

Larger $u$ values move faster, and thus eventually overtake smaller values. This ultimately results in the development of, for example, a right traveling shock if the initial data for $u$ is any positive, decreasing function. The speed at which a shock moves is not related in any simple way to the bulk flow speed or characteristic speed, and is not immediately evident from examining the flux, in contrast to contacts. Shock speed is controlled by the difference between influx and outflux of conserved quantity into the region. Specifically, suppose a conserved quantity $\phi$ with conservation law

$$\phi_t + f(\phi)_x = 0$$

has a step function profile with constant values extending both to the left, $\phi_L$, and to the right, $\phi_R$, with a single shock jump transition in between moving with speed $s$. Then the integral form of the conservation law,

$$\frac{d}{dt} \int_\Omega \phi \, dV = - \int_{\partial\Omega} \mathbf{f}(\phi) \cdot dS$$

applied to any interval containing the shock, gives the relation

$$\begin{aligned} \frac{s\Delta t(\phi_R - \phi_L)}{\Delta t} &= f(\phi_R) - f(\phi_L) \\ \Rightarrow s(\phi_R - \phi_L) &= f(\phi_R) - f(\phi_L) \end{aligned}$$

which is just another statement that the rate at which $\phi$ appears, $s(\phi_R - \phi_L)$, in the interval of interest is given by the difference in fluxes across the interval. See Figure 1.

We use the notation $[\phi]$ to denote the difference in $\phi$ across the control volume. The shock speed can then be written

$$s = \frac{[f(\phi)]}{[\phi]}$$

for the scalar conservation law or

$$s[\boldsymbol{\phi}] = [\mathbf{f}(\boldsymbol{\phi})]$$

for a system of conservation laws.

For example, consider Burgers' equation with $u_L = 1$ and $u_R = -1$. We compute the shock speed

$$\begin{aligned} f(u_L) &= \frac{u_L^2}{2} = \frac{1}{2} \\ f(u_R) &= \frac{u_R^2}{2} = \frac{1}{2} \\ s &= \frac{[f(u)]}{[u]} = \frac{0}{2} = 0. \end{aligned}$$

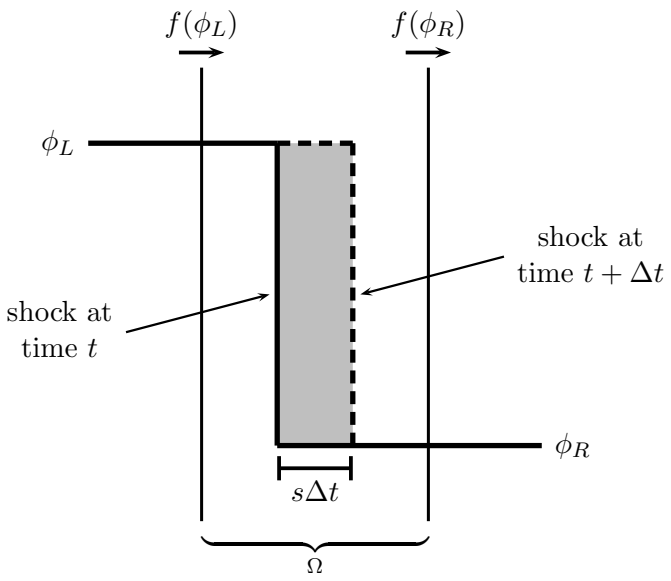


Figure 1: Derivation of the shock speed for a conserved quantity $\phi$ with a step function profile with constant values extending both to the left, $\phi_L$, and to the right, $\phi_R$. The shaded region is the additional amount of $\phi$ in the control region due to the motion of the shock.

The computation shows that the shock sits still. If we take $u_L = 2$ and $u_R = -1$, we have

$$\begin{aligned} f(u_L) &= \frac{u_L^2}{2} = \frac{4}{2} = 2 \\ f(u_R) &= \frac{u_R^2}{2} = \frac{1}{2} \\ s &= \frac{[f(u)]}{[u]} = \frac{\frac{1}{2} - 2}{-1 - 2} = \frac{-1.5}{-3} = \frac{1}{2}. \end{aligned}$$

Thus we see that the proper speed of the shock is directly determined by conservation of $u$ via the flux $f$. This has an important implication for numerical method design. Namely, a numerical method will only capture the correct shock speeds if it has conservation form, i.e. if the rate of change of $u$ at some node is the difference of fluxes which are accurate approximations of the real flux $f$. The Lax-Wendroff theorem tells us that if a consistent and conservative method converges, then it converges to a weak solution of the conservation law. Note that a weak solution may not be unique.

Shocks have a self-sharpening feature that has two implications for numerical methods. First, it means that even if the initial data is smooth, steep gradients and jumps will form spontaneously. Thus, our numerical method must be prepared to deal with shocks even if none are present in the initial data. Second, there is a beneficial effect from self-sharpening, because modest numerical errors introduced near a shock (smearing or small oscillations) will tend to be eliminated, and will not accumulate. The shock is naturally driven towards its proper shape. Because of this, computing strong shocks is mostly a matter of picking a conservative scheme that will correctly determine their speed.

## 1.4 Rarefactions

Whereas Burgers' equation with monotonically decreasing initial data results in the formation of a shock, Burgers' equation with monotonically increasing initial data results in the formation of a *rarefaction*. A rarefaction is a discontinuous jump or steep gradient in properties that dissipates as a smooth expansion. A common example is the jump in air pressure from outside to inside a balloon which dissipates as soon as the balloon is burst and the high pressure gas inside is allowed to expand. Such an expansion also occurs when the piston in an engine is rapidly pulled outward from the cylinder.

A rarefaction tends to smooth out local features, which is generally beneficial for numerical modeling. However, a rarefaction often connects to a smooth (e.g. constant) solution region and this results in a "corner" which is notoriously difficult to capture accurately. The main numerical problem posed by rarefactions is that of initiating the expansion. If the initial data is a perfect, symmetrical step, such as $u(x) = \text{sign}(x)$, it may be "stuck" in this form, since the steady state Burgers' equation is satisfied identically (i.e. the flux $u^2/2$ is constant everywhere, and similarly in any numerical discretization). However, local analysis can identify this stuck expansion, because the characteristic speed $u$ on either side points away from the jump suggesting its potential to expand. In order to get the initial data unstuck, a small amount of smoothing must be applied to introduce some intermediate state values that have a non-constant flux to drive expansion,

$$u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0.$$

In numerical methods, this smoothing applied at a jump where the effective local velocity indicates expansion should occur is called an *entropy fix*, since it allows the system to evolve from the artificial low entropy initial state to the proper increased entropy state of a free expansion, and the solution obtained is called the *vanishing viscosity* solution. This solution is the unique weak solution that satisfies the entropy condition.

Let us illustrate the point above. Assume we are solving Burgers' equation with initial data $u_L = -1$ and $u_R = 1$. Then

$$\begin{aligned} f(u_R) &= \frac{1}{2} \\ f(u_L) &= \frac{1}{2} \\ \Rightarrow u_t &= 0. \end{aligned}$$

In this case our numerical solution is "stuck", and we are computing an entropy-violating expansion shock. We don't want this solution. In order to obtain the correct, entropy condition satisfying solution (which is the rarefaction) we add some numerical smearing by solving

$$u_t + \left(\frac{u^2}{2}\right)_x = \epsilon u_{xx}$$

so that initially we are solving

$$u_t = \epsilon u_{xx}.$$

This smears out the step function profile enough to initiate the rarefaction.

One challenge in computing numerical solutions to hyperbolic conservation laws is that a wrong solution might look very good visually, but still be incorrect. For example, if our scheme is not in conservation form, the solution might look almost correct, except that the location of the shock will be off by a few grid cells.