CME306 / CS205B Homework 9

1. Write out the *symmetric* matrix equation for the standard second order central difference approximation to the equation

$$\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = \nabla \cdot \vec{u}^\star \tag{1}$$

with the following boundary conditions:

$$\begin{cases} p(0, y) = 1 \\ p(1, y) = 1 \\ p_y(x, y) = 0 \quad \text{for } y \in \{0, 1\} \end{cases} \tag{2}$$

You should assume a MAC grid (ie. that velocities live on cell faces, and that pressure and density live in the cell centers), and you may *not* assume a constant density. Write the equations for the following three cells:

(a) an internal cell (something sufficiently far from the boundary, ie. p_{ij})

$$\frac{\frac{2(p_{i+1,j}-p_{i,j})}{(\rho_{i,j}+\rho_{i+1,j})\Delta x} - \frac{2(p_{i,j}-p_{i-1,j})}{(\rho_{i-1,j}+\rho_{i,j})\Delta x}}{\Delta x} + \frac{\frac{2(p_{i,j+1}-p_{i,j})}{(\rho_{i,j}+\rho_{i,j+1})\Delta y} - \frac{2(p_{i,j}-p_{i,j-1})}{(\rho_{i,j-1}+\rho_{i,j})\Delta y}}{\Delta y}$$
$$= \frac{u^\star_{i+1/2,j} - u^\star_{i-1/2,j}}{\Delta x} + \frac{v^\star_{i,j+1/2} - v^\star_{i,j-1/2}}{\Delta y}$$

(b) a cell that lies along the x-axis boundary (ie. $p_{1,j}$), and

$$\frac{\frac{2(p_{2,j}-p_{1,j})}{(\rho_{1,j}+\rho_{2,j})\Delta x} - \frac{2p_{1,j}}{(\rho_{0,j}+\rho_{1,j})\Delta x}}{\Delta x} + \frac{\frac{2(p_{1,j+1}-p_{1,j})}{(\rho_{1,j}+\rho_{1,j+1})\Delta y} - \frac{2(p_{1,j}-p_{1,j-1})}{(\rho_{1,j-1}+\rho_{1,j})\Delta y}}{\Delta y}$$
$$= \frac{u^\star_{3/2,j} - u^\star_{1/2,j}}{\Delta x} + \frac{v^\star_{1,j+1/2} - v^\star_{1,j-1/2}}{\Delta y} - \frac{2}{(\rho_{1,j} + \rho_{2,j})\Delta x^2}$$

(c) a cell that lies along the y-axis boundary (ie. $p_{i,1}$).

$$\frac{\frac{2(p_{i+1,1}-p_{i,1})}{(\rho_{i,1}+\rho_{i+1,1})\Delta x} - \frac{2(p_{i,1}-p_{i-1,1})}{(\rho_{i-1,1}+\rho_{i,1})\Delta x}}{\Delta x} + \frac{\frac{2(p_{i,2}-p_{i,1})}{(\rho_{i,1}+\rho_{i,2})\Delta y}}{\Delta y} = \frac{u^\star_{i+1/2,1} - u^\star_{i-1/2,1}}{\Delta x} + \frac{v^\star_{i,3/2} - v^\star_{i,1/2}}{\Delta y}$$

2. Physically, when we have an incompressible flow with all Neumann boundary conditions, what does the compatibility condition require? Is something similar required for compressible flow?

Recall that the compatibility condition requires $\int_{\partial\Omega} \vec{u}^\star \cdot \vec{n} dS = 0$ *– or equivalently that the incompressible fluid flow is not being either* **compressed** *or* **expanded** *by the boundary. No such condition exists for compressible flow, since it is free to compress or expand as necessary.*