CME306 / CS205B Homework 7

ENO-LLF

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1}$$

Consider Burgers' equation (above), discretized in a conservative way with forward-Euler:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\mathcal{F}^n_{i+1/2} - \mathcal{F}^n_{i-1/2}}{\Delta x} = 0. \tag{2}$$

Recall from lecture (and section) that we compute $\mathcal{F}^n_{i+1/2}$ the numerical flux via a variety of ENO schemes.

1. Recall that Lax-Friedrichs defines, for flux $\mathcal{F}_{i_0+1/2}$, $D_i^1 H^{\pm} = f(\phi_i) \pm \alpha_{i_0+1/2}\phi_i$. Show that this is equivalent to adding some viscosity to the solution and that this viscosity vanishes as $\Delta t, \Delta x \to 0$.

   We examine the flux component of equation (1)*:*

$$\begin{aligned}
\frac{\mathcal{F}^n_{i+1/2} - \mathcal{F}^n_{i-1/2}}{\Delta x} &= \frac{(\mathcal{F}^{n+}_{i+1/2} + \mathcal{F}^{n-}_{i+1/2}) - (\mathcal{F}^{n+}_{i-1/2} + \mathcal{F}^{n-}_{i-1/2})}{2\Delta x} \\
&= \frac{f(u_i) + f(u_{i+1}) + \alpha_{i_0+1/2}(u_i - u_{i+1}) - f(u_{i-1}) - f(u_i) - \alpha_{i_0-1/2}(u_{i-1} - u_i)}{2\Delta x} \\
&= \frac{f(u_{i+1}) - f(u_{i-1}) + \alpha_{i_0+1/2}(u_i - u_{i+1}) - \alpha_{i_0-1/2}(u_{i-1} - u_i)}{2\Delta x} \\
&= \frac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x} - \Delta x \frac{\alpha_{i_0+1/2}(u_{i+1/2})_x - \alpha_{i_0-1/2}(u_{i-1/2})_x}{2\Delta x} \\
&= f(u_i)_x + \mathcal{O}(\Delta x^2) - \Delta x \frac{\alpha_{i_0+1/2}(u_{i+1/2})_x - \alpha_{i_0-1/2}(u_{i-1/2})_x}{2\Delta x}
\end{aligned}$$

   We note here that if we are using **global** *Lax-Friedrichs,* $\alpha_{i_0+1/2} = \alpha_{i_0-1/2} = \alpha$*, giving:*

$$\boxed{(u_i)_t + f(u)_x - \frac{\alpha\Delta x}{2}(u_i)_{xx} = \mathcal{O}(\Delta x^2)}$$

   In **local** *Lax-Friedrichs, we note that* $\alpha_{i_0+1/2} = \alpha_{i_0-1/2} + \mathcal{O}(\Delta x)$ *and we get a similar result:*

$$\boxed{(u_i)_t + f(u)_x - \frac{\alpha\Delta x}{2}(u_i)_{xx} = \mathcal{O}(\Delta x^2)}$$

2. In Local Lax-Friedrichs (or ENO-LLF), we take $\alpha_{i_0+1/2} = \max\{|\lambda_{i_0}|, |\lambda_{i_0+1}\}$. A common mistake (one that shows up even in the literature) is to try to compute a global divided difference table as

$$D_i^1 H^{\pm} = f(u_i) \pm \alpha_{i+1/2} u_i, \tag{3}$$

   Which is *very* different from the correct, **local** divided difference table for $\mathcal{F}_{i_0+1/2}$,

$$D_i^1 H^{\pm} = f(u_i) \pm \alpha_{i_0+1/2} u_i. \tag{4}$$

   There are advantages to computing a global divided difference table, by saving the need to recompute quantities that've already been computed. How can you do this for ENO-LLF while preserving the correct scheme?

The simplest approach would be to compute **two** *divided difference tables:*

$$D_i^1 H = f(u_i)$$

and

$$D_i^1 G = u_i.$$

Then we can take

$$D_k^n H^{\pm} = D_k^n H \pm \alpha_{i_0+1/2} D_k^n G \tag{5}$$

at each face. Since we already have the global divided difference tables, we only perform one multiplication for every lookup, rather than having to compute the divided difference table for every face.

Discrete Conservative Form

1. Consider the following discretizations of Burger's equation (1). Show that they are both consistent, but that only one of them is in conservative form. Which one is in conservative form?

   (a)

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_j^n)^2}{2\Delta x} = 0}$$

$$\begin{aligned}
0 &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_j^n)^2}{2\Delta x} \\
&= \frac{u_j^n + u_t \Delta t + \mathcal{O}(\Delta t^2) - u_j^n}{\Delta t} + \frac{\left(\frac{u^2}{2}\right)_{j+1}^n - \left(\frac{u^2)}{2}\right)_j^n}{\Delta x} \\
\Rightarrow & \boxed{0 = u_t + \left(\frac{u^2}{2}\right)_x + \mathcal{O}(\Delta t, \Delta x)}
\end{aligned}$$

   This is written in conservative form, and is consistent. To see that this is in conservation form, consider the following:

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N} u_i \Delta x &= \Delta x \sum_i \frac{u_i^{n+1} - u_i^n}{\Delta t} \\
&= \sum_i \left[\frac{{u_{i+1}^n}^2}{2} - \frac{{u_i^n}^2}{2}\right] \\
&= \left[\frac{{u_{N+1}^n}^2}{2} - \frac{{u_1^n}^2}{2}\right] \\
&= f(u_{N+1}^n) - f(u_1^2)
\end{aligned}$$

   (b)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

$$\begin{aligned}
0 &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} \\
&= u_t + \mathcal{O}(\Delta t) + u(u_x + \mathcal{O}(\Delta x)) \\
\Rightarrow & \boxed{0 = u_t + u u_x + \mathcal{O}(\Delta t, \Delta x)}
\end{aligned}$$

   This is consistent, but not in closed form. To see this, consider the following:

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N} u_i \Delta x &= \sum_i \frac{u_i^{n+1} - u_i^n}{\Delta t} \\
&= \sum_i u_i^n \left[u_{i+1}^n - u_i^n\right]
\end{aligned}$$

   Where the interior terms do **not** *cancel.*

2. The equations given above are both consistent with Burger's equation, but only one of them gives shocks traveling at the right speed. Why is this?

   The conservative formulation is guaranteed not to add or destroy material away from the interface, so the total amount of u-stuff that pushes the shocks around doesn't change non-physically. In the non-conservative formulation we attempt to approximate the derivative u_x between cell i and cell $i+1$; near shocks this derivative doesn't exist, and we have $\mathcal{O}(1)$ new material being added or removed; this changes how the shock moves, giving an incorrect shock speed.

3. Consider the simple advection equation (6). Show that for this equation, ENO-Roe and ENO-LLF schemes will produce identical results.

$$u_t + au_x = 0 \tag{6}$$

   it suffices to show that the flux computed by both methods agree exactly. To show this, observe that the advection equation has a flux $f(u) = au$. Then the divided difference tables for ENO-LLF take the form:

$$\begin{aligned} D_i^1 H^+ &= f(u_i) + |a|u_i = \begin{cases} 2au_i = 2D_i^1 H & a > 0 \\ 0 & a \leq 0 \end{cases} \\ D_i^1 H^- &= f(u_i) - |a|u_i = \begin{cases} 0 & a > 0 \\ 2au_i = 2D_i^1 H & a \leq 0 \end{cases} \end{aligned} \tag{7}$$

   In either case, one of the divided difference tables is uniformly zero, and the other divided difference table is exactly twice the magnitude of the divided difference table of ENO-Roe. The scaling by two doesn't change how ENO traverses the divided difference table, so either $\mathcal{F}_{Roe} = \mathcal{F}_{LLF}^+$ or $\mathcal{F}_{Roe} = \mathcal{F}_{LLF}^-$ depending on the sign of a. This gives that every flux $\mathcal{F}_{i_0+1/2}$ is exactly equivalent, and thus the two schemes give identical results.