CME306 / CS205B Homework 6

Essentially Non-Oscillatory Schemes

Given the following data for ϕ^n, write down the interpolating polynomial that third order HJ ENO would construct in order to compute ϕ_i^{n+1} in approximating the equation $\phi_t + \phi_x = 0$.

$$\phi^n_{i-3} = 5, \phi^n_{i-2} = 5, \phi^n_{i-1} = 4, \phi^n_i = 5, \phi^n_{i+1} = 1, \phi^n_{i+2} = -2, \phi^n_{i+3} = 0$$

Recall that the interpolating polynomial for 3^{rd} order requires Q_1, Q_2, Q_3; Q_0 will be calculated, but then promptly discarded since $(Q_0)_x = 0$. Next, we calculate the divided difference table, below:

$i-3$		$i-2$		$i-1$		i		$i+1$		$i+2$		$i+3$
5		5		4		5		1		-2		0
	0		$\frac{-1}{\Delta x}$		$\frac{1}{\Delta x}$		$\frac{-4}{\Delta x}$		$\frac{-3}{\Delta x}$		$\frac{2}{\Delta x}$	
		$\frac{-1}{2\Delta x^2}$		$\frac{1}{\Delta x^2}$		$\frac{-5}{2\Delta x^2}$		$\frac{1}{2\Delta x^2}$		$\frac{5}{\Delta x^2}$		
			$\frac{3}{6\Delta x^3}$		$\frac{-7}{6\Delta x^3}$		$\frac{6}{6\Delta x^3}$		$\frac{4}{6\Delta x^3}$			

We are evaluating ϕ_x at i, so $Q_0 = \phi_i = 5$. We required an upwind direction, which gives us Q_1, and ENO gives Q_2 and Q_3 as:

$$Q_1 = \frac{1}{\Delta x}(x - x_i)$$
$$Q_2 = \frac{1}{\Delta x^2}(x - x_i)(x - x_{i-1})$$
$$Q_3 = \frac{1}{2\Delta x^3}(x - x_i)(x - x_{i-1})(x - x_{i-2})$$

Putting it all together, we get:

$$\boxed{P^3(x) = 5 + \frac{1}{\Delta x}(x - x_i) + \frac{1}{\Delta x^2}(x - x_i)(x - x_{i-1}) + \frac{1}{2\Delta x^3}(x - x_i)(x - x_{i-1})(x - x_{i-2})} \quad (1)$$

We'll go a few steps further now, to find out what $\phi_x(x_i)$ approximately is. We evaluate $P^3_x(x_i)$ to be:

$$P^3_x(x) = \tfrac{1}{\Delta x} + \tfrac{1}{\Delta x^2}\left[(x - x_i) + (x - x_{i-1})\right] + \tfrac{1}{2\Delta x^3}\left[(x - x_i)\left[(x - x_{i-1}) + (x - x_{i-2})\right] + (x - x_{i-1})(x - x_{i-2})\right]$$
$$P^3_x(x_i) = \tfrac{1}{\Delta x} + \tfrac{1}{\Delta x^2}(x_i - x_{i-1}) + \tfrac{1}{2\Delta x^3}(x_i - x_{i-1})(x_i - x_{i-2})$$
$$= \tfrac{1}{\Delta x} + \tfrac{1}{\Delta x} + \tfrac{1}{\Delta x} = \boxed{\frac{3}{\Delta x}}$$

If we happened to have chosen that $\Delta x = .5$, then $\phi_x \approx 6$.

Weighted ENO

If we consider an upwind discretization of ϕ_x, we have three possible third-order interpolating polynomials, given by

$$\begin{aligned}
\phi_x^1 &= \frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6} \\
\phi_x^2 &= -\frac{v_2}{6} + \frac{5v_3}{6} + \frac{v_4}{3} \\
\phi_x^3 &= \frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6}
\end{aligned}$$

Where $v_j = D^*\phi_{i+j-3}$, and $D^*\phi$ is the first-order upwind discretization of ϕ_x.

However, the philosophy of picking exactly one of the three candidate stencils is overkill in smooth regions of ϕ where ϕ is well-behaved. Instead, we can take a convex sum of the three stencils,

$$\phi_x = \omega_1\phi_x^1 + \omega_2\phi_x^2 + \omega_3\phi_x^3 \tag{2}$$

Where $0 \leq \omega_i \leq 1$, $\omega_1 + \omega_2 + \omega_3 = 1$. It has been shown that we can pick $\omega_1 = .1, \omega_2 = .6, \omega_3 = .3$ and achieve a 5^{th} order accurate approximation of ϕ_x.

1. Show that if we perturb ω by $\mathcal{O}(\Delta x^2)$ we still get a 5^{th} order approximation to ϕ_x.

   we know that each of ϕ_x^j for $j \in \{1,2,3\}$ are third-order accurate schemes, so $\phi_x^j = \phi_x + \mathcal{O}(\Delta x^3)$. If we take $\epsilon_j = \mathcal{O}(\Delta x^2)$ to be our perturbations to ω_j, then our WENO scheme for ϕ_x becomes:

$$\begin{aligned}
\phi_x &= \bar{\omega}_1\phi_x^1 + \bar{\omega}_2\phi_x^2 + \bar{\omega}_3\phi_x^3 \\
&= (\omega_1 + \epsilon_1)\phi_x^1 + (\omega_2 + \epsilon_2)\phi_x^2 + (\omega_3 + \epsilon_3)\phi_x^3 \\
&= \omega_1\phi_x^1 + \omega_2\phi_x^2 + \omega_3\phi_x^3 + \epsilon_1\phi_x^1 + \epsilon_2\phi_x^2 + \epsilon_3\phi_x^3 \\
&= \phi_x + \mathcal{O}(\Delta x^5) + (\epsilon_1 + \epsilon_2 + \epsilon_3)\phi_x + \epsilon_1\mathcal{O}(\Delta x^3) + \epsilon_2\mathcal{O}(\Delta x^3) + \epsilon_3\mathcal{O}(\Delta x^3) \\
&= \phi_x + (\epsilon_1 + \epsilon_2 + \epsilon_3)\phi_x + \mathcal{O}(\Delta x^5)
\end{aligned}$$

   We note that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ since we still want $\sum_j \bar{\omega}_j = 1$, and this scheme is 5^{th} order accurate.

2. Why is this a bad idea in non-smooth areas of the flow? In order to demonstrate this, consider $\phi_t + \phi_x = 0$ for a heaviside step function, with initial data given by:

$$\phi_{i-3}^n = 0, \phi_{i-2}^n = 0, \phi_{i-1}^n = 0, \phi_i^n = 1, \phi_{i+1}^n = 1, \phi_{i+2}^n = 1, \phi_{i+3}^n = 1$$

   We've discussed in class that any scheme which adds over-shoots to a problem can lead to non-physical oscillations near discontinuities. With that in mind, consider the WENO approximation which is made for ϕ_x at x_{i-1}. The divided difference table takes the form:

$i-4$		$i-3$		$i-2$		$i-1$		i		$i+1$		$i+2$		$i+3$
0		0		0		0		1		1		1		1
	0		0		0		$\frac{1}{\Delta x}$		0		0		0	

   If we read off the table, we get:

$$v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0 \qquad v_4 = \frac{1}{\Delta x} \qquad v_5 = 0$$

   Both ϕ_x^2 and ϕ_x^3 give a non-zero approximation to ϕ_x, even though both the ENO approximation as well as the analytical solution gives $\phi_{i-1} = 0$ for $t > 0$. In HJ-WENO there is **no way** *to avoid pulling in bad information near a discontinuity, which is why it is not a good method to use near non-smooth regions of the flow.*