CME306 / CS205B Homework 5

Mass-Spring stress-strain relationship

Consider the mass-spring model discussed in class. In many ways, this is just a specific constitutive model for solids – as such, there should be a way to write the stress-strain relationship explicitly, given an appropriate choice of measure of strain. This would show that mass-spring models are a subset of the material properties that can be modeled via finite elements. For this problem, we'll consider a two-dimensional triangle and derive some of the finite element components for this mass-spring system.

1. The Cauchy strain is a measure of strain that is invariant to a number of operations... Briefly explain why this measure is an *inappropriate* choice of strain for the constituative model of our mass-spring system.

   Cauchy strain is invariant to linear rotations, mass-spring systems are not.

2. Sketch the triangle in world space (labeling the edge-lengths as ℓ_1, ℓ_2, ℓ_3), then write down the force contributions (f_1, f_2, f_3) from the triangle acting on nodes x_1, x_2 and x_3 respectively. You should assume that the springs are *not damped*.

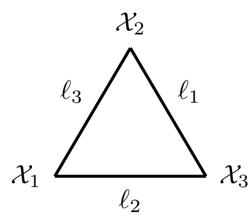

Figure 1: An equilateral triangle in world space, with a suspiciously uniform deformation :-)

   Each of $\vec{f_1}, \vec{f_2}$ and $\vec{f_3}$ can be written as a linear combination of the forces exerted by the three springs. We define the following vectors: $\vec{v}_1$ is the vector from x_2 to x_3, $\vec{v}_2$ is the vector from x_3 to x_1, and $\vec{v}_3$ is the vector from x_1 to x_2. This gives the forces as:

$$\vec{f_1} = k\left[-(\frac{\ell_2}{\ell_{2_0}} - 1)\frac{\vec{v}_2}{\ell_2} + (\frac{\ell_3}{\ell_{3_0}} - 1)\frac{\vec{v}_3}{\ell_3}\right]$$
$$\vec{f_2} = k\left[(\frac{\ell_1}{\ell_{1_0}} - 1)\frac{\vec{v}_1}{\ell_1} - (\frac{\ell_3}{\ell_{3_0}} - 1)\frac{\vec{v}_3}{\ell_3}\right]$$
$$\vec{f_3} = k\left[-(\frac{\ell_1}{\ell_{1_0}} - 1)\frac{\vec{v}_1}{\ell_1} + (\frac{\ell_2}{\ell_{2_0}} - 1)\frac{\vec{v}_2}{\ell_2}\right]$$

3. These forces represent different linear combinations of the traction that appears on each face of the triangle. Write down the equations for these forces in terms of the Cauchy stress tensor (which is a matrix of four unknowns) and the area-weighted normals.

   The force contribution from the triangle to the node is given by:

$$\vec{f_1} = -\frac{1}{2}\left(\sigma\vec{\mathcal{N}_2} + \sigma\vec{\mathcal{N}_3}\right) = -\frac{1}{2}\sigma\left(\vec{\mathcal{N}_2} + \vec{\mathcal{N}_3}\right)$$
$$\vec{f_2} = -\frac{1}{2}\left(\sigma\vec{\mathcal{N}_1} + \sigma\vec{\mathcal{N}_3}\right) = -\frac{1}{2}\sigma\left(\vec{\mathcal{N}_1} + \vec{\mathcal{N}_3}\right)$$
$$\vec{f_3} = -\frac{1}{2}\left(\sigma\vec{\mathcal{N}_1} + \sigma\vec{\mathcal{N}_2}\right) = -\frac{1}{2}\sigma\left(\vec{\mathcal{N}_1} + \vec{\mathcal{N}_2}\right)$$

To find the area weighted normal, we can observe that $\vec{\mathcal{N}_i} \cdot \vec{v}_i = 0$ *by definition, and that* $\|\vec{v}_i\| = \ell_i$. *If* $\vec{v}_i = (v_{i1} \quad v_{i2})^T$, *then* $\vec{\mathcal{N}_i}$ *can be written as* $(v_{i2} \quad -v_{i1})^T$.

4. Solve these equations to get σ.

   Examining the equations given in part 4 we can write down the following equations, where the columns of $\mathbf{N}$ *are defined as* $\frac{\vec{\mathcal{N}_2}+\vec{\mathcal{N}_3}}{2}$ *and* $\frac{\vec{\mathcal{N}_1}+\vec{\mathcal{N}_3}}{2}$, *and the columns of* $\mathbf{F}$ *are defined as* $\vec{f_1}$ *and* $\vec{f_2}$ *respectively.*

$$\mathbf{N} = \begin{pmatrix} \frac{\vec{\mathcal{N}_2}+\vec{\mathcal{N}_3}}{2} & \frac{\vec{\mathcal{N}_1}+\vec{\mathcal{N}_3}}{2} \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} \vec{f_1} & \vec{f_2} \end{pmatrix}$$

$$\Rightarrow \boxed{\sigma = \mathbf{F}\mathbf{N}^{-1}}$$

   Note that $\mathbf{N}$ *is invertible except when* $\vec{\mathcal{N}_1}+\vec{\mathcal{N}_3} = \vec{\mathcal{N}_2}+\vec{\mathcal{N}_3}$ *or* $\vec{\mathcal{N}_1}+\vec{\mathcal{N}_3} = -(\vec{\mathcal{N}_2}+\vec{\mathcal{N}_3})$, *which happens when* $\vec{\mathcal{N}_3} = 0$ *or when* $\vec{\mathcal{N}_3}$ *is parallel to both* $\vec{\mathcal{N}_1}$ *and* $\vec{\mathcal{N}_2}$. *In either of these cases, the triangle has collapsed into a line.*

   Note that

$$\mathbf{N}^{-1} = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{22} & -n_{12} \\ -n_{21} & n_{11} \end{pmatrix}$$

   So

$$\boxed{\sigma = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} f_{11}n_{22} - f_{12}n_{21} & f_{12}n_{11} - f_{11}n_{12} \\ f_{21}n_{22} - f_{22}n_{21} & f_{22}n_{11} - f_{21}n_{12} \end{pmatrix}} \tag{1}$$