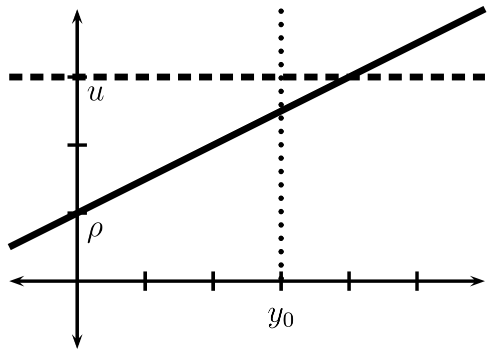
Conservation of Mass (Eulerian Framework)

1. In an Eulerian framework, the strong form of Conservation of Mass takes the form below. Please briefly explain the three nonzero terms in the equation.

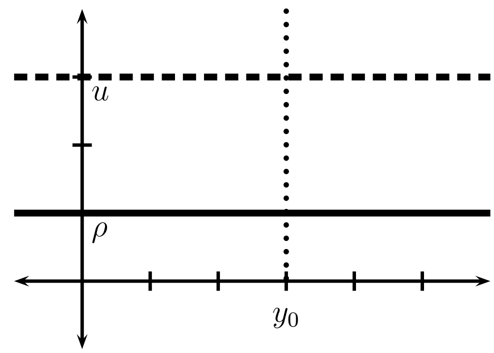
$$\rho_t + \rho u_x + u\rho_x = 0 \tag{1}$$

2. If we are working with a discontinuous density field or velocity field (i.e. either ρ_x or u_x don't exist somewhere in the domain), we cannot use the strong form of Conservation of Mass. We *can* however apply the weak form, which describes how mass changes in a control volume Ω (here, mass is given as $\int_\Omega \rho dV$). Please derive the weak form equation for Conservation of Mass from the strong form (note that the weak form should have no spatial derivatives).

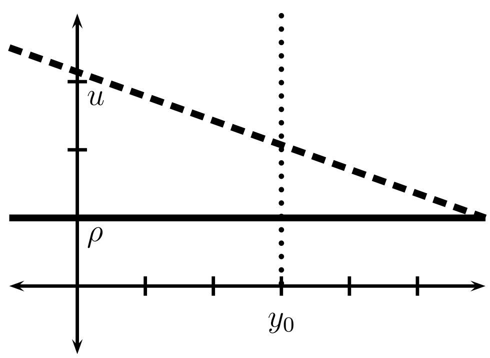
3. In an Eulerian framework, the graphs of ρ and u in the plots below describe the state of a system. Does ρ increase, decrease, or stay the same at the sample point y_0 for each system? (You may assume all quantities here are positive)



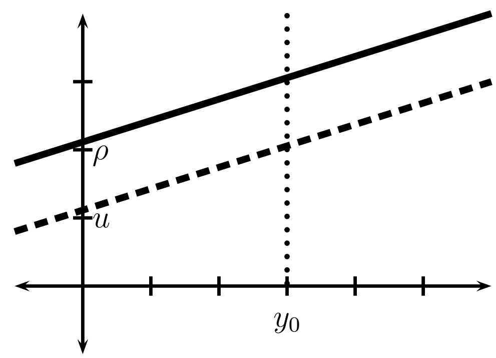
(a) System 1



(b) System 2



(c) System 3



(d) System 4

Convergence Analysis

Consider the wave equation

$$u_t + au_x = 0$$

where $a = constant$. Establish whether or not the following methods for solving the equation converge. If so, what are the conditions for convergence? **Hint:** Use the Lax-Richtmyer equivalence theorem. Chapters 1 and 2 of the text by Strikwerda will be helpful, in addition to the discussion notes provided online.

Note that $(D^+\phi)_i = \frac{\phi_{i+1}-\phi_i}{\Delta x}$, and $(D^-\phi)_i = \frac{\phi_i-\phi_{i-1}}{\Delta x}$.

1. Explicit Central Differencing
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = 0.$$
2. Implicit Central Differencing
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x} = 0.$$
3. Upwinding
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + aD^*v_j^n = 0$$
If $a > 0$, $D^* = D^-$. If $a < 0$, $D^* = D^+$.
4. Downwinding
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + aD^*v_j^n = 0$$
If $a > 0$, $D^* = D^+$. If $a < 0$, $D^* = D^-$.