CME306 / CS205B Extra Credit 2 (Theory Track)

Consider the Poisson equation:

$$\begin{cases} \nabla \cdot \frac{1}{\rho} \nabla p = \nabla \cdot \mathbf{u}^\star & \text{in } \Omega \\ p = p_0(x, y) & \text{on } \partial\Omega_1 \\ \mathbf{u} \cdot \mathbf{n} = u_0(x, y) & \text{on } \partial\Omega_2 \end{cases} \tag{1}$$

Where $\partial\Omega_1$ is the region whose boundary conditions are specified as *Dirichlet*, and $\partial\Omega_2$ is the region whose boundary conditions are specified as *Neumann*. Note that $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$, the entire boundary of the domain.

Create a 64×64 MAC grid for $\Omega = (0, 1) \times (0, 1)$. Fill the velocity field with random numbers and treat these values as $\mathbf{u}^\star$; then solve the Poisson equation (with your choice of boundary conditions). Apply the resulting pressure to $\mathbf{u}^\star$ in order to get a divergence-free velocity field, and submit plots of the initial condition, the pressure, and the resulting velocity field. Do this for 4 different velocity fields, with a variety of choices for the boundary conditions.

You can (and should) take ρ to be constant. You must consider several different choices of $\partial\Omega_1$ and $\partial\Omega_2$, including at least one which has $\partial\Omega_1 = \emptyset$ (i.e. the boundary must be all Neumann). You are encouraged to use Matlab, which provides the backslash operator "\" – this makes solving a linear system of equations $Ax = b$ to be a simple call: "$x = A\backslash b$".

Submit these plots (there should be 12 in total), along with a short (one to two page) description of your implementation, and your sourcecode. In your write-up, you should prove that the boundary conditions on $\partial\Omega_2$ are equivalent to first *setting* $\mathbf{u}^\star \cdot \mathbf{n}|_{\partial\Omega_2} = u_0|_{\partial\Omega_2}$ and then solving the Poisson equation with $\nabla \cdot p = 0$.