Convergence, Consistency, and Stability

Definition
A one-step finite difference scheme approximating a partial differential equation is a ***convergent*** scheme if for any solution to the partial differential equation, $u(t,x)$, and solutions to the finite difference scheme, v_i^n, such that v_i^0 converges to $u_0(x)$ as $i\Delta x$ converges to x, then v_i^n converges to $u(t,x)$ as $(n\Delta t, i\Delta x)$ converges to (t,x) as $\Delta t, \Delta x$ converge to 0.

Definition
Given a partial differential equation $Pu = f$ and a finite difference scheme, $P_{\Delta t,\Delta x} v = f$, we say that the finite difference scheme is ***consistent*** with the partial differential equation if for any smooth function $\phi(x,t)$

$$P\phi - P_{\Delta t,\Delta x}\phi \to 0 \text{ as } \Delta t, \Delta x \to 0.$$

Example
Consider the one-way wave equation given by the operator $P = \partial/\partial t + \alpha\partial/\partial x$

$$P\phi = \phi_t + \alpha\phi_x$$

with α greater than 0. We will evaluate the consistency of the forward-time forward-space scheme with difference operator $P_{\Delta t,\Delta x}$ given by

$$P_{\Delta t,\Delta x}\phi = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \alpha\frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} \tag{1}$$

where

$$\phi_i^n = \phi(n\Delta t, i\Delta x).$$

We begin by taking the Taylor expansion of the function ϕ in t and x about (t_n, x_i). We have that

$$\phi_i^{n+1} = \phi_i^n + \Delta t\phi_t + \frac{1}{2}\Delta t^2\phi_{tt} + O(\Delta t^3)$$

$$\phi_{i+1}^n = \phi_i^n + \Delta x\phi_x + \frac{1}{2}\Delta x^2\phi_{xx} + O(\Delta x^3).$$

This gives us

$$P_{\Delta t,\Delta x}\phi = \phi_t + \alpha\phi_x + \frac{1}{2}\Delta t\phi_{tt} + \alpha\frac{1}{2}\Delta x\phi_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

Thus

$$\begin{aligned} P\phi - P_{\Delta t,\Delta x}\phi &= \tfrac{1}{2}\Delta t\phi_{tt} + \tfrac{1}{2}\Delta x\phi_{xx} + O(\Delta t^2) + O(\Delta x^2) \\ &\to 0 \quad \text{as} \quad (\Delta t, \Delta x) \to 0. \end{aligned}$$

Thus, this scheme is consistent. □

Definition
The $L2$-norm of a grid function w, denoted by $\|w\|_{\Delta x}$, is defined as

$$\|w\|_{\Delta x} = \left(\Delta x \sum_{m=-\infty}^{\infty} |w_m|^2 \right)^{1/2}.$$

Definition
A finite difference scheme $P_{\Delta t, \Delta x} v_i^n = 0$ for a first-order equation is **stable** in a stability region Λ if there is an integer J such that for any positive time T, there is a constant C_T such that

$$\|v^n\|_{\Delta x} \leq C_T \sum_{j=0}^{J} \|v^j\|_{\Delta x}$$

for $0 \leq n\Delta t \leq T$, with $(\Delta t, \Delta x) \in \Lambda$.

Von Neumann Analysis

Proving stability directly from the definition is quite difficult, in general. Instead, it is easier to use tools from Fourier analysis to evaluate the stability of finite difference schemes. In particular, it can be shown that, for some solution to a finite difference scheme v^n, there is a simple mathematical relationship between the Fourier transforms $\hat{v}^n(\xi)$ and $\hat{v}^0(\xi)$ given by

$$\hat{v}^n(\xi) = g(\Delta x \xi, \Delta t, \Delta x)^n \hat{v}^0(\xi)$$

where $g(\Delta x \xi, \Delta t, \Delta x) = g(\theta, \Delta t, \Delta x)$ is called the *amplification factor* and $\hat{v}^n(\xi)$ is the amplitude of the frequency ξ in the solution v^n (note that the second superscript n in the above equation is a power and not an index). This quantity is so named because it represents the amount that each frequency in the solution is amplified in advancing the solution one time step. That this relationship arises from such analysis should not be surprising: the fundamental power of Fourier transforms is that they transform differentiation in the temporal domain to multiplication in the frequency domain.

Definition
A one-step finite difference scheme with constant coefficients is stable in a stability region Λ if and only if there is a constant K (independent of θ, Δt, and Δx) such that

$$|g(\theta, \Delta t, \Delta x)| \leq 1 + K\Delta t$$

with $(\Delta t, \Delta x) \in \Lambda$. If $g(\theta, \Delta t, \Delta x)$ is independent of Δt and Δx, the stability condition may be replaced with the restricted stability condition

$$|g(\theta)| \leq 1.$$

Example
Consider again the one-way wave equation and let us evaluate the stability of the forward-time forward-space scheme given in equation (1). It can be shown that all solutions of any one-step difference scheme will have the form $v_j^n = g^n e^{ij\theta}$. To analyze the stability of such a scheme, we may therefore substitute accordingly and solve for g.

$$\frac{g^{n+1}e^{ij\theta} - g^n e^{ij\theta}}{\Delta t} + \alpha \frac{g^n e^{i(j+1)\theta} - g^n e^{ij\theta}}{\Delta x} = 0$$

$$\frac{g^n e^{ij\theta}(g-1)}{\Delta t} = -\alpha \frac{g^n e^{ij\theta}(e^{i\theta}-1)}{\Delta x}$$

$$(g-1) = -\alpha(e^{i\theta}-1)\frac{\Delta t}{\Delta x}$$

$$g = 1 + \alpha\frac{\Delta t}{\Delta x} - \alpha\frac{\Delta t}{\Delta x}e^{i\theta}$$

$$g = 1 + \alpha\lambda - \alpha\lambda e^{i\theta}$$

where $\lambda = \Delta t/\Delta x$ and α is positive. We can then calculate

$$|g|^2 = 1 + 4\alpha\lambda(1+\alpha\lambda)\sin^2\frac{1}{2}\theta.$$

Since λ is constant, we may use the restricted stability condition and we see that $|g|$ is greater than 1 for $\theta \neq 0$. Therefore, this scheme is unstable. □

The Lax-Richtmyer Equivalence Theorem

The Lax-Richtmyer Equivalence Theorem is often called the *Fundamental Theorem of Numerical Analysis*, even though it is only applicable to the small subset of linear numerical methods for well-posed, linear partial differential equations. Along with Dahlquist's equivalence theorem for ordinary differential equations, the notion that the relationship

$$\text{consistency} + \text{stability} \iff \text{convergence}$$

always holds has caused a great deal of confusion in the numerical analysis of differential equations. In the case of PDEs, mathematicians are most often interested in nonlinear phenomena, for which Lax-Richtmyer does not apply. More damningly, the forward implication that consistency + stability $\implies$ convergence is trivial for linear schemes, and thus it is only the converse notion that convergence $\implies$ stability that the theorem contributes. The intuition that the theorem gives for problems that fall outside the scope of Lax-Richtmyer, however, is faulty, since consistency and stability are often insufficient for convergence, and convergence need not imply stability in general.

Suppose we have a stable, linear numerical method that is consistent with a well-posed linear partial differential equation. Suppose further that the global error at time $T = N\Delta t$ is the grid function denoted by

$$E^N = Q^N - q^N$$

where Q^n is the numerical approximation to the solution and q^n is the exact solution $\forall n \leq N$. We will denote our method by $\mathcal{N}(\cdot)$, such that

$$Q^{n+1} = \mathcal{N}(Q^n).$$

We claim that it is straightforward to show that the method is convergent, which amounts to proving that $\|E^N\| \to 0$ as $\Delta t \to 0$ and $N\Delta t \to T$.

Since $\mathcal{N}$ is stable, we know that there exists a constant C_T depending only on T such that

$$\|\mathcal{N}^n\| \leq C_T, \qquad \forall n \leq N = T/\Delta t.$$

We can also express the local trunctation error as the one step error divided by the time step, or

$$\tau^n = \frac{1}{\Delta t}\left[\mathcal{N}(q^n) - q^{n+1}\right],$$

and note that consistency implies that $\tau^n \to 0$ as $\Delta t \to 0$.

We can derive a simple recurrence for the error at time $n+1$:

$$\begin{aligned}
E^{n+1} &= Q^{n+1} - q^{n+1} \\
&= \mathcal{N}(Q^n) - q^{n+1} \\
&= \mathcal{N}(E^n + q^n) - q^{n+1} \\
&= \mathcal{N}(E^n + q^n) - \mathcal{N}(q^n) + \mathcal{N}(q^n) - q^{n+1} \\
&= \mathcal{N}(E^n + q^n) - \mathcal{N}(q^n) + \triangle t\, \tau^n \\
&= \mathcal{N}(E^n) + \triangle t\, \tau^n
\end{aligned}$$

where we use the linearity of $\mathcal{N}(\cdot)$ in the last equality. From this formulation, it is simple to examine the norm of the error and argue that the requirements for convergence are met by stability and consistency.