CME 192: Introduction to MATLAB
Lecture 6

Stanford University

January 29, 2019

Outline

Review

Ordinary Differential Equations

Solving Equations

Function Approximation

Review

Lecture 5

- Timing
- Optimization
  - Preallocation
  - **Vectorization**
  - Using in-built functions
  - Memory layout
- Error Handling

Outline

Ordinary Differential Equations

Ordinary

- one independent variable (usually time, t)
- **cannot** be time and length (heat flow problem)

Differential

- $x', \dot{x}, \frac{dx}{dt}$
- $x'', \ddot{x}, \frac{d^2x}{dt^2}$
- $x''', \dddot{x}, \frac{d^3x}{dt^3}$

Equations

- $\dot{x} = f(t, x)$
- e.g. $\dot{x} = -x^2 + t$;
- multiple equations are OK

Dynamics Equations

Dynamics Equations are Ordinary Differential Equations

$$ma = F$$

$$v = \dot{x}$$

$$a = \dot{v} = \ddot{x}$$

so

$$\ddot{x} = \frac{F}{m} = \frac{1}{m} F(t, x)$$

is an Ordinary Differential Equation

$$\ddot{x} = f(t, x)$$

Solving Ordinary Differential Equations

- choose starting point (initial conditions)
- advance in time, for example:

$$x(0) = x_0$$

$$x(t + \Delta t) = x(t) + \Delta t \cdot f(t, x(t))$$

- repeat till desired time is reached
- more accurate methods exist

$$f(t, x) = x$$

$$\Delta t = 0.1$$

```
x(0.0) = x0 = 1
x(0.1) = 1 + 0.1 * 1.0 = 1.1
x(0.2) = 1.1 + 0.1 * 1.1 = 1.21
x(0.3) = 1.21 + 0.1 * 1.21 = 1.33
x(0.4) = 1.33 + 0.1 * 1.33 = 1.46
x(0.5) = 1.46 + 0.1 * 1.46 = 1.61
```

First Order Ordinary Differential Equations

$$\dot{x} = x^2 + t \quad \Longrightarrow \quad \begin{bmatrix} \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_1^2 + t \end{bmatrix}$$

$$\ddot{x} = x^2 + t \quad \Longrightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1^2 + t \end{bmatrix}$$

Notice that

$$\frac{d}{dt} x_2 = \dot{x}_2 = \frac{d}{dt} \dot{x}_1 = \ddot{x}_1$$

$$\ddddot{x} = \dot{x} - x^2 + t \quad \Longrightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_2 - x_1^2 + t \end{bmatrix}$$

Solving First Order Ordinary Differential Equations

Procedure

1. write the function $f(t, x)$
2. choose time span on which to solve
   (just start and end points are OK)
3. choose initial conditions (of x)
4. run a differential equation solver

```
% 1. write the function
f = @(t, x) -x^2 + t;
% 2. time span
% (doesn't affect accurracy)
tspan = linspace(0, 10, 1e3);
% 3. initial conditions
x0 = 0;

% 4. run solver
[T, X] = ode45(f, tspan, x0);
```

Outline

Finding a zero of a function

- Zeros of a polynomial
  - `<roots> = roots(<poly_coeff>)`, e.g. `r = roots([2, 3, 1])`
  - always works, gives complex roots too
- Zeros of a univariate $f(x) = 0$ function
  - `<x_zero> = fzero(<fn>, <x_guess>)`
  - doesn't always work, function has to change sign at zero
  - solves: $x^2 - 2 = 5x + 2 \implies f(x) = x^2 - 2 - 5x - 2 = 0$
- System of equations $\begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
  - `<x_zero> = fsolve(<Fn>, <x_guess>)`
  - takes a function `Fn = @(X)` where `Fn` is a vector of functions and `X` is a vector of variables
  - doesn't always work, function has to change sign at zero

Solving Systems of Linear Equations

$$\begin{cases} 3x + 5y + z = 0 \\ 7x - 2y + 4z = 2 \\ -6x + 3y + 2z = -1 \end{cases} \implies \begin{bmatrix} 3 & 5 & 1 \\ 7 & -2 & 4 \\ -6 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & 1 \\ 7 & -2 & 4 \\ -6 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \implies \begin{aligned} Ax &= b \\ x &= A^{-1}b \end{aligned}$$

$$x = A^{-1}b \implies$$

```
A = [3, 5, 1; ...
     7, -2, 4; ...
     -6, 3, 2]
b = [0; 2; -1];
% A x = b
x = inv(A) * b;
```

Solving Systems of Linear Equations

$$Ax = b$$

Matrix Inverse

```
x = inv(A) * b
```

- unique solution must exist (gives garbage otherwise)
- same number of equations and unknowns

Matrix Pseudoinverse

```
x = pinv(A) * b
```

- if matrix is invertible, same answer as `inv`
- if matrix is not invertible
  - if too many equations: smallest total error
  - if too few equations: smallest vector that satisfies equations

Backslash

```
x = A \ b
```

- very advanced, chooses best algorithm

Outline

Finding a line between points

$$f(x) = mx + b$$

Between two points

$$\begin{cases} mx_1 + b = y_1 \\ mx_2 + b = y_2 \end{cases}$$

solving for m and b. In matrix form

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Between more points

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Best fit

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

has no unique solution. Try to find such m and b that error is smallest

$$\left|\left| \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \right|\right| = \text{error} = ||A\theta - y||$$

```
th = A \ y
```

Quadratic fit

$$f(x) = ax^2 + b + c$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \\ 1 & x_6 & x_6^2 \\ 1 & x_7 & x_7^2 \\ 1 & x_8 & x_8^2 \\ 1 & x_9 & x_9^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ \vdots \\ y_n \end{bmatrix}$$

More complex functions

$$
\begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 & \sin(x_1) & e^{x_1} \\
1 & x_2 & x_2^2 & x_2^3 & \sin(x_2) & e^{x_2} \\
1 & x_3 & x_3^2 & x_3^3 & \sin(x_3) & e^{x_3} \\
1 & x_4 & x_4^2 & x_4^3 & \sin(x_4) & e^{x_4} \\
1 & x_5 & x_5^2 & x_5^3 & \sin(x_5) & e^{x_5} \\
1 & x_6 & x_6^2 & x_6^3 & \sin(x_6) & e^{x_6} \\
1 & x_7 & x_7^2 & x_7^3 & \sin(x_7) & e^{x_7} \\
1 & x_8 & x_8^2 & x_8^3 & \sin(x_8) & e^{x_8} \\
1 & x_9 & x_9^2 & x_9^3 & \sin(x_9) & e^{x_9} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_n & x_n^2 & x_n^3 & \sin(x_n) & e^{x_n}
\end{bmatrix}
\begin{bmatrix}
\theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ \vdots \\ y_n
\end{bmatrix}
$$