Lockbox Separation

William F. Sharpe
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Introduction

This note develops the concept of *lockbox separation* for retirement financial strategies in a complete market. I show that in such a setting any strategy can be implemented by dividing initial wealth among a series of "lockboxes", each designed to fund spending at a particular date using a predetermined investment strategy for managing the funds until that date. Such an approach allows a retiree to pre-commit to follow spending and investment rules throughout the remainder of his or her life. This may have significant advantages since the decisions can be made prior to future reductions in reasoning ability, cognition, etc.. In effect, the current person can act in *loco parentis* for a potentially diminished future person.

To keep the exposition simple I omit issues associated with additional sources of income, mortality, annuities, health insurance etc.. The approach can, however, be extended to cover many of these and other aspects of the real-world problems associated with financial planning in retirement.

The approach utilizes the formulation developed in my on-line textbook, *Macro-Investment Analysis*. For details, see the section on Dynamic Strategies, (last revised in April 1999) at www.wsharpe.com/mia/prc/mia_prc3.htm#dynamic.

A Binomial Example

For simplicity I utilize a three-date example in which security returns follow an independent and identical binomial process. In particular, at each date there are two one-period securities. A riskless bond (B) has a total real return of 1.02 – that is, for each dollar invested at a date the security provides 1.02 real dollars at the next date. A security invested in all the risky securities in the market (S) provides a total real return of 0.94 if the market is "down" and 1.18 if the market is "up". These two states are equally likely.

The retiree has W0 dollars to finance spending at the three dates. There are seven possible states of the world, shown by the following tree.

0 → u → uu
0 → u → ud
0 → d → du
0 → d → dd
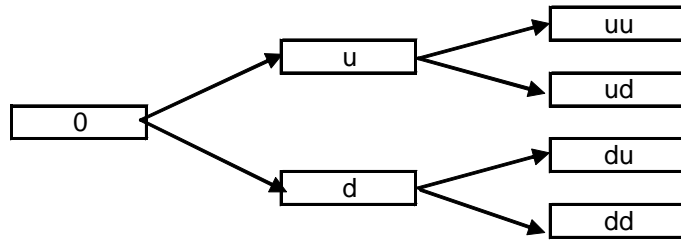
Retirement Financial Plans

A *retirement financial plan* assigns an amount to be spent (consumed) at each node on such a tree. The amounts to be spent must be obtained by investing W0 dollars in available capital market instruments. The retiree's goal is to maximize the expected utility of the amounts to be spent subject to the constraint that the total cost equal the initial wealth W0.

The capital market conditions can be summarized in a square matrix **M** which will have as many rows and columns as there are states. The matrix for this example is shown below.

	W0	B0	S0	Bd	Sd	Bu	Su
0	1	-1	-1	0	0	0	0
d	0	1.02	0.94	-1	-1	0	0
u	0	1.02	1.18	0	0	-1	-1
dd	0	0	0	1.02	0.94	0	0
du	0	0	0	1.02	1.18	0	0
ud	0	0	0	0	0	1.02	0.94
uu	0	0	0	0	0	1.02	1.18

Each row represents a state, or node on the tree. The columns represent decision variables. W0 can be considered a security that pays $1 at the initial date. B0 represents a decision to invest $1 at date 0 in a riskless bond, which will provide $1.02 at date 1 whether the market is up or down. S0 represents a decision to invest $1 in the risky security at date 0, which will pay $0.94 at date 1 if the market is down and $1.18 if the market is up. Bd represents a decision to invest in a riskless bond at date 1 if the market is down at that time. It will provide payments at date 2 in the two states that are subsequently possible. The remaining variables are similar, with each representing a decision made initially to take an action contingent on the occurrence of a particular state of the world.

A financial plan can be characterized in two ways. The first is represented by a vector **x**, which shows the values assigned to each of the decision variables. For example[1]:

W0	100.00
B0	1.96
S0	59.68
Bd	1.32
Sd	25.70
Bu	1.64
Su	31.92

This plan requires an initial wealth of $100, since it calls for the purchase of 100 units of security W0, which costs $1 per unit. However, no additional wealth is required, since each of the other decision variables represents a "zero-investment strategy". For example, the value of 1.96 for B0 indicates that $1.96 is to be invested at date 0 in riskless bonds. This will result in a payment of 1.02*1.96 at date 1 if the market is down or a payment of 1.02*1.96 at date 1 if the market is up. No initial funds are required. Each of the other decision variables has this property. In general, the cost (initial wealth required) for a financial plan will equal x(1).

The other way to characterize a financial plan focuses on the amounts to be spent in each possible state. We denote this vector **c** (for consumption). For the plan shown by x, the amounts spent will be as follows:

0	38.36
d	31.08
u	38.86
dd	25.51
du	31.67
ud	31.67
uu	39.34

Clearly there is a close relationship between **x** and **c**. Importantly:

$$Mx = c$$

Thus capital market conditions (**M**) as well as wealth and the contingent decisions to be taken (**x**) determine consumption (**c**). The process can also be reversed. Assume that one wishes to obtain a particular set of amounts to be consumed (**c**). Then the required decision variables (**x**) will given by:

$$x = M^{-1}c$$

Where M^{-1} is the inverse of M.

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[1] All values in this paper are rounded to the number of decimal places shown.

For this example, M^{-1} (rounded to four places) is:

1.0000	0.6536	0.3268	0.4272	0.2136	0.2136	0.1068
0.0000	4.8203	-3.8399	3.1505	1.5752	-2.5097	-1.2549
0.0000	-4.1667	4.1667	-2.7233	-1.3617	2.7233	1.3617
0.0000	0.0000	0.0000	4.8203	-3.8399	0.0000	0.0000
0.0000	0.0000	0.0000	-4.1667	4.1667	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	4.8203	-3.8399
0.0000	0.0000	0.0000	0.0000	0.0000	-4.1667	4.1667

There is an important economic meaning associated with each column of this matrix. Consider a desired plan to spend $1 in state u. Vector **c** will thus be:

0	0.00
d	0.00
u	1.00
dd	0.00
du	0.00
ud	0.00
uu	0.00

The required set of decision variables is:

W0	0.3268
B0	-3.8399
S0	4.1667
Bd	0.0000
Sd	0.0000
Bu	0.0000
Su	0.0000

This is the third column of the inverse of M. It shows the cost of a "state claim" to receive $1 in state u and nothing otherwise is $0.3268. This is defined as the "state price" for the state. The remainder of the column shows the strategy required to obtain the desired outcome: buy $4.1667 of stocks and sell $3.8399 of bonds. The net cost of these two transactions is, of course, $0.3268.

Each column of the inverse of M provides the cost of a state claim (in the top row) and the strategy that can produce the desired payment.

Our example is one in which there is a complete market, in the sense that it is possible to obtain any one of N state claims by following a strategy using currently available securities. Key to this is the existence of a set of N decision variables using available securities that provide payments in an NxN matrix that can be inverted.

Expected Utility Maximization

The consumption amounts **c** used in our example were not chosen casually. Rather, they represent an optimal solution for a retiree who wishes to maximize expected utility, given an initial wealth of 100, in the manner described in my working paper *Retirement Financial Planning: A State Preference Approach* (February 2007) at www.wsharpe.com/wp/rfp/pdf. In this case the utility functions were time-varying with constant relative risk aversion at each date. Specifically, for each date, $U = a(C^{1-g})/(1-g))$ where the values were as follows:

Date	a	g
0	1.00	3.00
1	0.98	3.10
2	0.96	3.20

The optimal values for **c** for such an investor can be found using the state prices obtained from the inverse of matrix **M**, as described in the referenced working paper. Then the required decision variables can be found by multiplying the inverse of M by vector **c**.

An alternative approach uses the matrix **M** directly. Construct an excel spreadsheet with **M**, **x** and **c**, where **c** is obtained by multiplying **M** by **x**. For every value in **c** provide a formula that computes the associated utility. Finally, multiply each utility by the probability of the associated state and sum the products to obtain the overall expected utility. The goal is to find a vector **x** with the initial value equal to initial wealth that will maximize expected utility. This can be accomplished using Excel's *solver* program (available as an add-in). The resulting vectors **c** and **x** will provide the overall investment and spending rules constituting the optimal retirement financial plan. If the optimization is successful[2] the results will be the same as those obtained using the prior procedures.

Lockbox Separation

I turn now to the issue of lockbox separation. Recall the relationship between the vector of desired consumption c and the associated financial strategy:

$$x = M^{-1}c$$

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[2] It may be necessary to experiment with the solver settings to obtain sufficient precision and avoid numeric problems.

Now, divide **c** into a number of desired consumption vectors c_0, c_1 and c_2, which sum to the overall vector **c**:

$$c = c_0 + c_1 + c_2$$

The optimal strategy x will be:

$$x = M^{-1} (c_0 + c_1 + c_2)$$

or:

$$x = M^{-1}c_0 + M^{-1}c_1 + M^{-1}c_2$$

But this can be implemented using a series of strategies x_0, x_1 and x_2 where:

$$x_i = M^{-1}c_i$$

This relationship can be used for any partition of c but our interest is in a partition in which all the states at a given date are included in one such vector. For our example:

	C0	C1	C2
0	38.36	0.00	0.00
d	0.00	31.08	0.00
u	0.00	38.86	0.00
dd	0.00	0.00	25.51
du	0.00	0.00	31.67
ud	0.00	0.00	31.67
uu	0.00	0.00	39.34

The required strategies are:

	X0	X1	X2
W0	38.36	33.01	28.63
B0	0.00	0.57	1.40
S0	0.00	32.44	27.23
Bd	0.00	0.00	1.32
Sd	0.00	0.00	25.70
Bu	0.00	0.00	1.64
Su	0.00	0.00	31.92

Each of these can be implemented with a separate lockbox.

The first lockbox will begin with $38.36, which will be used for immediate consumption.

The second lockbox will begin with $33.01 which will be used to purchase $0.57 of bonds and $32.44 of stocks. At date 1 the box will be opened. If the market is down there will be $31.08 to spend; if the market is up there will be $38.86.

The third lockbox will begin with $28.63 that will be used to purchase $1.40 of bonds and $27.23 of stocks. At date 1 the portfolio will be changed. If the market is down, the holdings will be revised so that $1.32 is invested in bonds and $25.70 in stocks. If the market is up the holdings will be altered so that $1.64 is invested in bonds and $31.92 in stocks. When the box is opened at date 2 there will be $25.51, $31.67 or $39.34 to be spent, depending on the course of the market moves up to that date.

Asset Allocations

It is straightforward to compute the percentage of total value invested in stocks at each time and state for any given strategy. For, example, at date 0 lockbox 1 has $32.44 invested in stocks and $0.57 in bonds. Thus 98.28% or 0.9828 is invested in stocks. Lockbox 2 has 95.12% (0.9512) invested in stocks at date 0 and also at date 1, whether the market is up or down. This is not surprising, since in our setting maximization of expected utility with a constant relative risk aversion function requires a constant asset allocation in value terms.

It is important to note, however, that the overall asset allocation does change through time. Performing the same calculations for the vector c shows that the overall percentage invested in stocks is initially 96.81%, falling to 95.12% at date 1, whether the market is down or up. This is not surprising, since the overall asset allocation can be thought of as a type of average of the asset allocations in the remaining unspent lockboxes. Since the retiree's risk-aversion is greater for consumption at later dates, lockboxes designed for later consumption have lower percentages of stock. As time goes on, the remaining lockboxes collectively provide a more conservative overall investment strategy.

Extensions and Applications

Our example has utilized a highly simplified model of security returns, with only two states of the world at each date as well as security returns that are the same in each period. Moreover, for our expected utility optimization we assumed that up and down moves were equally probable and that each of the retiree's utility functions exhibited constant relative risk aversion. Some of our results were dependent on one or more of these assumptions. However, the lockbox separation property is quite general. As long as the capital market is complete, our basic equation relating M, x and c will hold. This in turn will permit separation of an overall optimal investment strategy into a series of strategies, one to finance spending for each date, with each such strategy implemented in

a lockbox to be used for that date, with an initial amount invested and a designated investment strategy to be followed until the date in question.

But why utilize a lockbox strategy? An overall strategy with the appropriate spending and investment rules will provide the same outcomes. The answer hinges on assumptions about the ability of the retiree to make optimal decisions in later years. Consider, for example, a person who is now 65 years old. She could leave investment and spending decisions 20 years hence to her "future self". But there is a possibility that at that point she might not be capable of making decisions that are truly in her best interests. One alternative would be to have a wise counselor make her financial decisions at the time. Another would be for her "present self" to make decisions now for her "future self." A lockbox strategy would facilitate this. As a practical matter, of course, there would almost certainly need to be provisions that would allow changes if certain unanticipated events transpired. But it could be advantageous to make plans now, when one has one's faculties, that at least specify appropriate reactions to future market moves.

For these reasons it appears that additional research in this area is warranted. Possible extensions could cover mortality, insurance vehicles, incomplete markets and transactions costs (broadly construed). Applications could investigate typical retiree preferences and other relevant behavioral aspects. Clearly, more research is needed.