CME 305: Discrete Mathematics and Algorithms

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Problem Session #1 – 02/11/15

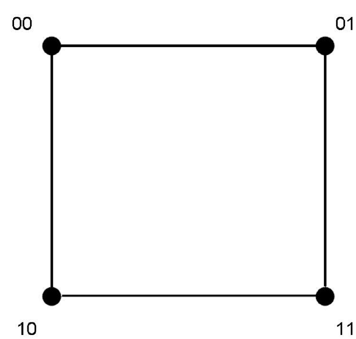
1. **(Kleinberg Tardos 11.10)** Suppose you are given an n by n grid graph G with vertex weights $w(v) \geq 0$ that are all distinct and integer. The goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible.

   Consider the "heaviest-first" greedy algorithm. Start with $S = \emptyset$, and while $|V| \neq 0$ pick the node $v_i \in V$ of maximum weight, add v_i to S and delete its neighbors from G. The resulting S will be an independent set by construction.

   (a) Let S be the independent set returned by the algorithm above, and let T be any other independent set in G. Show that, for each node $v \in T$, either $v \in S$, or there is a node $v' \in S$ so that $w(v) \leq w(v')$ and $(v, v') \in E$.

   (b) Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.

2. An n-dimensional cube can be represented by a graph with 2^n vertices with every vertex corresponding to an n-bit binary number. Two vertices are connected by an edge if their corresponding binary numbers differ by only one bit. For example, the following represents a 2-D cube.

   Prove that every n-dimensional cube has a Hamiltonian cycle.

3. **(Lovasz, Pelikan, and Vesztergombi 8.5.6)** A *double star* is a tree that has exactly two nodes that are not leaves. How many unlabeled double stars are there on n nodes?

4. **(Lovasz, Pelikan, and Vesztergombi 8.5.10)** If C is a cycle, and e is an edge connecting two nonadjacent nodes of C, then we call e a *chord* of C. Prove that if every node of a graph G has degree at least 3, then G contains a cycle with a chord.

5. Show that every graph $G = (V, E)$ has a subgraph on at least $|E|/2$ edges which is bipartite.

6. Suppose $G = (V, E)$ has degree sequence d_1,...,d_n, where $n = |V|$. Show that G has an independent set of size at least:

$$\sum_{j=1}^{n} \frac{1}{d_j + 1}$$

Hint: consider a random permutation $\pi(*)$ of the vertices and consider the set: $A = \{x \in V | \pi(x) < \pi(v), \forall y \in N(x)\}$.

7. Let G be an undirected simple graph on n vertices and with m edges. Given a pair of vertices (s, t) design a random walk based algorithm to determine whether the two are connected. Your algorithm should must store only the current position of the random walk and have polynomial running time.

8. Show that every instance of 3-SAT has assignment of variables that satisfy at least a 7/8 fraction of the clauses.