Tranching and Rating

Michael J. Brennan
Julia Hein
Ser-Huang Poon *

March 10, 2008

*Michael Brennan is at the Anderson School, UCLA and the Manchester Business School, Email: michael.brennan@anderson.ucla.edu. Julia Hein is at the University of Konstanz, Department of Economics, Box D 147, 78457 Konstanz, Germany, Email: Julia.Hein@uni-konstanz.de. Ser-Huang Poon is at Manchester Business School, Crawford House, University of Manchester, Oxford Road, Manchester M13 9PL, UK. Tel: +44 161 275 0431, Fax: +44 161 275 4023, Email: ser-huang.poon@mbs.ac.uk.

Abstract

In this paper we analyze the gains to an investment banker who is able to market debt securities at yields that reflect the credit ratings of bond ratings agencies when the ratings depend on either the probabilities of default or the expected default losses of the securities issued. We consider the gains both from choosing the collateral against which the debt securities are written, and from dividing the debt into tranches with different priority. We derive general results and characterize the gains for numerical examples that are based on the CAPM and the Merton (1974) debt pricing model.

1 Introduction

Approximately $471 billion of the $550 billion of collateralized debt obligations (CDOs) issued in 2006, was classified by the Securities Industry and Financial Markets Association (SIFMA) as 'Arbitrage CDOs',[1]which are defined by SIFMA as an 'attempt to capture the mismatch between the yields of assets (CDO collateral) and the financing costs of the generally higher rated liabilities (CDO tranches).'[2] In the simple world of Modigliani and Miller (1958) such arbitrage opportunities could not exist.

In this paper we present a simple theory of the effect of collateral diversification and the tranching of debt contracts on the prices at which the debt securities can be marketed. The theory, which can account for the apparent arbitrage opportunities offered by the market for CDOs and other structured products,[3] rests on the assumption that some investors are not able to assess the value of the securities themselves, but must rely instead on the bond ratings provided by third parties. Such an assumption is justified by casual empiricism. We shall make the extreme assumption that securities can be sold in the primary market at yields which reflect only their ratings. This is not to say that all investors rely only on bond ratings - but that at least some do, and that if ratings based valuations exceed fundamental values, then the investment banker will be able to sell to these investors in the primary market at prices that depend only on ratings. The importance of ratings in the marketing of tranched securities has been widely noted in the media in the past year, and it has even been suggested, although this is denied by the ratings agencies themselves, that ratings agencies assist in the design of new securities to ensure that they achieve targeted rating assessments.[4] An article in the *Financial Times* of December 6, 2007 writes that 'for many investors ratings have served as a universally accepted benchmark', and that 'some funds have rued their heavy dependence on ratings'.

We do not argue that the marketing story we tell is the only explanation for the tranching of debt contracts. Previous contributions rely on asymmetric information and ability of the issuer either to signal the quality of the underlying assets by the mix of securities sold,[5] or on the differential ability of investors to assess complex risky securities. Thus, in Boot and Thakor (1993) cash flow streams are marketed by dividing them and allocating the resulting components to information insensitive and sensitive (intensive) securities. The former

[1]The remaining issuance is classified as 'Balance Sheet' CDOs which 'remove assets or the risk of the assets off the balance sheet of the originator'.

[2]SIFMA, January 2008. $http://archives1.sifma.org/assets/files/SIFMA_CDOIssuanceData2007q1.pdf$

[3]Grinblatt and Longstaff (2000), discuss arbitrage opportunities in the markets for stripped treasury securities.

[4]The ambiguities in the relation between the issuer and the rating agency are captured in a publication of Standard & Poor's: 'Either an issuer or an investment bank as the arranger presents a proposed structure. The rating analysts give their preliminary views as to what the rating will be,based upon our published criteria. The arranger in response may change aspects of the transaction. On unusual or novel types of transactions, this process may involve additional dialogue...It's important to re-iterate that in no way what occurs in the structured finance ever amount to "advisory" work.' Standard and Poor's (2007)

[5]Brennan and Kraus, 1987, De Marzo and Duffie 1999, DeMarzo 2005.

are marketed to uninformed investors, and the latter to information gathering specialist firms who face an exogenously specified deadweight cost of borrowing.

Our analysis is concerned with the defects of a bond rating system which relies only on assessments either of default probabilities or of expected losses due to default. It is straightforward to show that a system which relies only on default probabilities is easy to game - by selling securities with lower recovery rates than the securities on which the ratings are based. Only slightly more subtly, a system which relies on expected default losses is also easy to game. This is because a simple measure of expected default loss takes no account of the states of the world in which the losses occur. The investment banker may profit then by selling securities whose default losses are allocated to states with the highest state prices per unit of probability.[6]

We assume that the underlying asset against which the debt claims are written is properly valued by the investment banker. We also assume that bond ratings are calibrated with respect to single debt claims issued against a 'standard firm', by which we mean a firm with pre-specified risk characteristics. In this context we show that under a rating system that is based on default probabilities (e.g. Standard & Poor's and Fitch) the optimal strategy is to maximize the number of differently rated debt tranches. If the risk characteristics of the assets can be chosen, then they will be chosen to have the maximum beta and the minimum idiosyncratic risk. A rating system that is based on expected losses (e.g. Moody's) reduces some, but does not eliminate all, of the pricing anomalies and the investment banker's profit.

Our paper is related to the literature on the classical problem of capital structure. In an early contribution, Stiglitz (1972) stresses that the mix of securities sold may change the prices of all securities in the market by the (partial) completion of markets through the provision of a new security, whose payoffs are not spanned by existing securities. Short sales constraints offer another avenue by which the mix of financing may have a direct effect on valuation. As a simple example, consider a market in which investors differ in their assessment of the risk of the entity being financed. By selling debt securities to investors who have low risk assessments and equity securities to investors who have high risk assessments the investment banker may obtain a higher price for the entity than if equity alone is sold.

Ross (1989) has drawn attention to the marketing role of the investment banker...

Our analysis is most closely related to that of Coval et. al (2007) who show that it is possible to exploit investors, who rely on default probability based ratings for pricing securities, by selling 'catastrophe bonds', which are bonds whose default losses occur in high marginal utility states. They also calibrate a structural bond pricing model to observed CDX prices and use the model to predict yield spreads on CDX index tranches. Comparing these derived spreads to the spreads observed in the market, they conclude that there is severe market mispricing: the market spreads are much too low for the risk of the tranches. In contrast to Coval et. al, we explicitly derive yields on corporate bonds and on

[6] Coval et al (2007) make a similar point.

equally rated CDO-tranches within a simple theoretical framework. This allows us to explore the reasons for differences in yield spreads in more detail. We first examine the effects of diversification of the asset pool against which the bonds are written, and then the effects of issuing multiple tranches of debt securities. Fina;;y, we consider the effects of diversification and tranching jointly. We also consider, not only the case in which the rating is determined by the default probabilities, but also the case in which ratings are based on expected default losses.

Tranching of debt securities has been carried furthest in the recently developed structured credit markets in which collateralized debt securities (CDO's), collateralized loan obligations (CLO's), collateralized mortgage obligations (CMO's) and other asset backed securities (ABS) are routinely divided into tranches. These securities differ mainly in the nature of the underlying securities which are used to form the diversified portfolios against which the tranched securities are written. The fact that tranching is most often seen when the underlying assets are a diversified portfolio of securities is consistent with our finding that it is optimal for the investment bank to write the tranched securities against a portfolio with the highest possible component of systematic risk.

An important implication of the fact that tranched securities are typically written against diversified portfolios of securities is that defaults of tranched securities of a specified rating will tend to be much more highly correlated than defaults of securities of the same rating issued by a typical undiversified firm - in the limit the defaults of the tranched securities will be perfectly correlated. This, together with the systematic event of a decline in underwriting standards, accounts for the fact that we see almost all highly rated securities issued against portfolios of subprime mortgages made in 2006 and 2007 experiencing ratings deterioration at the same time. This has profound implications for regulatory systems for bank capital that depend on bond ratings.[7] A portfolio of n A rated CLO tranches will in general be much more risky than a portfolio of n A rated bonds issued by corporations. According to Standard & Poor's (2005) historical data indicates that rated tranches exhibit higher cumulative default probabilities and higher correlations than do corporate issues.[8]

Section 2 presents a general analysis of the investment banker's problem of security design and characterizes his arbitrage profit. Section 3 develops our valuation framework which is based on the CAPM and the Merton model of debt pricing. In section 4 we discuss several numerical examples, which illustrate how an investment banker can generate quite substantial profits by selling tranches at bond yields.

[7]Under Basel 1 the regulatory capital requirement was independent of the creditworthiness of the borrower. Under Basel II capital requirements depend either on external ratings, as discussed here, or on an approved internal rating system, which takes default probabilities and expected losses in case of default into account.

[8]SeeXXX???? rated firms credit curves (p.16) versus CDO tranches credit curves (p.19) and the correlation assumptions (p. 22) which are approximately twice as high as for ABS as compared to corporate issues.

2 Structured Bonds

In 1970 the Government National Mortgage Association (GNMA) sold the first securities backed by a portfolio of mortgage loans. In subsequent years GNMA further developed these securitisation structures through which portfolios of commercial or residential mortgages were sold to outside investors. From the mid 1980s this concept was transfered to other asset classes such as auto loans, corporate loans, corporate bonds, credit card receivables, etc. Since then the market for so called asset backed securities (ABS) has seen tremendous growth. According to the Bank of England (2007) the global investment volume in the ABS market was USD 10.7 trillion by the end of 2006.

In a securitization transaction a new legal entity, a Special Purpose Vehicle (SPV), is created to hold a designated portfolio of assets. The SPV is financed by a combination of debt and equity securities. A key feature is the division of the liabilities into tranches of different seniority: payments are made first to the *senior* tranches, then to the *mezzanine* tranches, and finally to the *junior* tranches. This prioritization scheme causes the tranches to exhibit different default probabilities and different expected losses: while the super-senior tranche is almost safe, the junior tranches bear the highest default risk.[9]

Typically the SPV issues two to five rated tranches and one non-rated equity or first loss piece (FLP). In an empirical study of European securitization transactions, Cuchra and Jenkinson (2005) found that a rather high percentage of the total portfolio volume is sold in tranches with a rating of A or better (on average 77%) and that AAA tranches on average accounted for 51% of the transaction but with a high variation across transactions types (between 30% and 89%). As shown in Franke *et al.* (2007) the size of the FLP varies significantly across transactions - from 2% and 20% in their sample.

The originator of the CDO specifies in advance the number of tranches and their desired ratings. Due to information asymmetries between the originator and the investors concerning the quality of the underlying portfolio, the tranches need to be rated by an external rating agency. After a thorough analysis of the transaction, which is mainly based on cashflow simulations and stress testing[10], two or three of the leading rating agencies assign ratings to the tranches. These ratings reflect the tranches' default probability (Standard & Poor's and Fitch) respectively expected loss (Moody's) and are used by investors as an indicator of the tranche's quality.

3 Credit Ratings

In the United States, there are seven rating agencies that have received the Nationally Recognized Statistical Rating Organization (NRSRO) designation,

[9]This is only a very brief and simplified description of these transactions. For a more detailed discussion on securitisation structures see Hein (2007)

[10]Beside this quantitative analysis, which plays a major role in the rating process, rating agencies also take qualitative aspects - including an analysis of the servicer's, asset manager's and trustee's skills and reputation as well as legal aspects - into account when rating a transaction.

and are overseen by the SEC: Standard & Poor's , Moody's, Fitch, A. M. Best, Japan Credit Rating Agency, Ltd., Ratings and Investment Information, Inc. and Dominion Bond Rating Service. S&P, Moody's, and Fitch dominate the market with approximately 90-95 percent of world market share. Amongst credit market participants, it is well known that Moody's ratings are based on their estimates of the expected losses due to default, while S&P and Fitch base their ratings on their estimates of the probability that the issuing entity will default.[11]

Standard and Poor's ratings for structured products have broadly the same default probability implications as their ratings for corporate bonds.[12] Prior to 2005 the implied default probabilities for corporate and structured product ratings were the same. In that year corporate ratings were "delinked from CDO rating quantiles" in order to avoid "avoid potential instability in high investment-grade scenario loss rates". Now, "CDO rating quantiles are higher than the corporate credit curves at investment grade rating levels, and converge to the corporate credit curves at low, speculative-grade rating levels".[13] Thus, S&P liberalised the ratings for structured bonds. Table 1 shows cumulative default frequencies for corporate bonds by rating and maturity as reported by Standard and Poor's (2005) and Table 2 shows the cumulative default frequency for CDO tranches. For example the five year cumulative default probability implied by a B rating for a CDO tranche is now 26.09 percent as compared with 24.46 percent for a corporate bond.

Moody's ratings for both corporate and structured bonds are based on the cumulative 'Idealized Loss Rates' which are shown in Table 3. Although it would seem more reasonable to base credit ratings on expected default losses rather than simply on default probability, Cuchra (2005, p 16) reports that in European markets for structured finance 'S&P ratings explain the largest share of the total variation in (new issue) spreads, followed by Moody's and Fitch.'

4 Arbitrage Gains from Securitization and Tranching

Among the primary roles of the investment banker are the marketing of new issues of securities, and the provision of advice on the appropriate mix of securities to finance a given bundle of assets. In the simple world of Modigliani and Miller (1958) there is no role for the investment banker since all financing mixes are equally good. However, since the pioneering contribution of MM it has been recognized that the mix of securities sold may be important for valua-

[11] Fender and Kiff (2004). S&P explicitly state that 'Our rating speaks to the likelihood of default, but not the amount that may be recovered in a post-default scenario.' Standard and Poor's (2008).

[12] For Standard and Poor's at least, the rating assigned to a particular tranche does not depend upon the size of the tranche, but only on the total face value of the tranche and tranches that are senior to it:'"Tranche thickness" generally does not affect our ratings, nor their volatility, since our ratings are concerned with whether or not a security defaults, *not how much loss it incurs in the event of default.*' Standard and Poor's (2007).

[13] See Standard & Poor's (2005).

tion on account of control, incentive, tax, and bankruptcy cost considerations. However, none of these factors offers any *direct* connection between the mix of securities and the valuation of given cash flow streams.

4.1 A Simple Model of Ratings Based Pricing

The importance of credit ratings for the pricing of structured bonds is documented by Cuchra (2005) who shows that 'the relation between price and credit rating for each tranche is very close indeed and consistent across all types of securitisations.. this relationship seems considerably stronger than in the case of corporate bonds.' This motivates our fundamental assumption that investment bankers are able to sell new issues of CDO tranches at yields to maturity which are the same as the yields on correspondingly rated corporate bonds.[14] The main difference between these two types of security is that the corporate bond is secured by the assets of a single firm and represents a senior claim, whereas the CDO-tranche is secured by a portfolio of bonds and is typically subordinated to higher rated tranches. For simplicity we will assume that the portfolio underlying the CDO consists of bonds issued by different small- and medium-sized firms and is homogeneous and granular. Additionally we assume that both securities have the same maturity τ.[15]

We shall use an asterisk to denote variables that correspond to the reference corporate bond or its issuer and use the same variables without the asterisk to denote the corresponding variable for the CDO or its SPV issuer. Thus let $W_k^*(V)$ and $W_k(V)$ denote the values of a pure discount debt security with face value B_k^* or B_k, rating k, and maturity τ when issued by a corporation and an SPV. Of course, the values depend on the value of the underlying assets of the corporate issuer or SPV, V^* and V.

Let $\phi_k^* \equiv W_k^*(V)/B_k^*$ denote the ratio of the market value of a corporate bond with rating k to its face value, and let $S_k(V)$ denote the *sales price* of a pure discount debt security with nominal value B_k and rating k issued by a Special Purpose Vehicle with initial value V. Then, our assumption is that the *sales price*, at which a new debt security issued by a Special Purpose Vehicle can be sold, bears the same relation to its face value as does the value of an equivalently rated debt security with the same maturity issued by a corporation:

Pricing Assumption:

$$S_k = \phi_k^* B_k.$$

Let P^* denote the (physical) probability distribution of the value of the corporation at the maturity of the bond, and let P denote the corresponding probability distribution for a given SPV. It is well-known that the price of any

[14] This assumption also seems to be consistent with the expectations of the rating agencies. For example, 'Do ratings have the same meaning across sectors and asset classes? The simple answer is "yes". Across corporates, sovereigns and structured finance,we seek to ensure to the greatest extent possible that the default risk commensurate with any rating category is broadly similar.' Standard and Poor's (2007).

[15] For simplicity we will drop the maturity subscript τ in the following.

contingent claim written on the value of the corporation, V^*, or the value of the SPV, V, can be expressed as the discounted value of the contingent claim payoff under the equivalent martingale measures Q^* and Q. The link between the physical and risk neutral measures is given by the conditional pricing kernels for contingent claims on the underlying assets, $m^*(v)$ and $m(v)$, with $f_{Q^*}(v) = m^*(v) f_{P^*}(v)$ and $f_Q(v) = m(v) f_P(v)$, and $f(v)$ is the density function of the terminal underlying asset value v under the corresponding measure.

We consider two different rating systems:

(i) *Default Probability Based Rating*

The bond rating, k, is a monotone decreasing function of the probability of default, $\mathcal{R}_{\mathcal{P}}(\Pi)$, $\mathcal{R}'_{\mathcal{P}}(\Pi) < 0$.

(ii) *Expected Default Loss Based Rating*

The bond rating, k, is a monotone decreasing function of the expected default loss, $\mathcal{R}_{\mathcal{L}}(\Lambda)$, $\mathcal{R}'_{\mathcal{L}}(\Lambda) < 0$.

We assume for simplicity that all defaults take place at maturity, and denote the default loss rate for a bond with rating k and maturity τ, by Λ_k, and denote the probability of default by Π_k.

The probabilities of default and the expected default loss rates are determined by the *physical* probability distributions, P and P^*, while the market values of the instruments, and therefore the ratios of market value to the nominal payments, are determined by the promised nominal payments and the risk neutral probability distributions, Q and Q^*, as illustrated below:

Agency Rated Corporate Bond:

$$\Lambda_k, \Pi_k \quad \stackrel{P_k^*}{\longleftarrow} \quad B_k^* \quad \stackrel{Q_k^*}{\longrightarrow} \quad \frac{W_k^*}{B_k^*} \equiv \phi_k^*$$

Agency Rated SPV Bond:

$$\Lambda_k, \Pi_k \quad \stackrel{P_k}{\longleftarrow} \quad B_k \quad \stackrel{Q_k}{\longrightarrow} \quad \frac{W_k}{B_k} \equiv \phi_k$$

Thus the fair *market value* of the SPV liability is:

$$W_k = \phi_k B_k$$

which usually differs from the ratings based *sales price* as defined before. In effect, we assume that the investment banker is able to sell the security at a price that reflects the risk neutral probability distribution, Q_k^*, that is appropriate for a typical corporate issuer of a bond with the same probability of default or expected loss.

First we consider the gains from securitization and tranching within a general model of valuation. In our subsequent analysis we shall present quantitative estimates from a parametric model of the arbitrage gains that the investment banker can reap from (i) differences between the physical probability distributions of the firms on which the ratings are based and the physical probability distributions of the SPV issuers; (ii) differences between the risk neutral probability distributions; (iii) using tranched security issues when there are different physical or risk neutral distributions.

4.2 A Single Tranche Securitization

As a preliminary, we characterize the arbitrage gain from securitising assets by issuing a single bonds against them. When ratings are based on *default probability*, the face value of the corporate bond with rating k is defined by $F_{P^*}(B_k^*) = \Pi_k$, or

$$B_k^* = F_{P^*}^{-1}(\Pi_k)$$

where F_{P^*} denotes the *cdf* corresponding to the physical probability measure P^*.

When ratings are based on *expected default loss*, the face value of a corporate bond with rating k is defined by $\int_0^{B_k^*}(B_k^* - v) f_{P^*}(v)dv = \Lambda_k$, where $f_{P^*}()$ is the density corresponding to $F_{P^*}()$. This implies

$$B_k^* = G_{P^*}^{-1}(\Lambda_k)$$

where $\int_0^{B_k^*}(B_k^* - v) f_{P^*}(v)dv \equiv G_{P^*}(B_k^*)$. Similarly the face value of the (single) debt tranche or CDO issued by the Special Purpose Vehicle to achieve the same rating is given by:

$$B_k = F_P^{-1}(\Pi_k)$$

in the case of default probability based rating, and by:

$$B_k = G_P^{-1}(\Lambda_k)$$

in the case of expected default loss based rating.

We may also express B_k^*, the nominal value of a corporate bond with a rating of k, in terms of B_k, the nominal value of an SPV bond with the same rating, under either a default probability or an expected default loss based rating system:

Under a default probability rating system:

$$B_k^* = F_{P^*}^{-1}[F_P(B_k)] \tag{1}$$

and under an expected default loss system:

$$B_k^* = G_{P^*}^{-1}[G_P(B_k)] \tag{2}$$

The arbitrage gain , $\mathbf{\Omega}$, from issuing the security is equal to the difference between the sales price, S_k, and the market value W_k:

$$\begin{aligned} \mathbf{\Omega} &= S_k - W_k \\ &= [\phi_k^* - \phi_k]\, B_k \end{aligned} \tag{3}$$

Setting the interest rate equal to zero for simplicity, the value of the security issued by the SPV is given by:

$$\begin{aligned} W_k &= \int_0^{B_k} v f_Q(v)dv + B_k \int_{B_k}^{\infty} f_Q(v)dv \\ &\equiv \phi_k B_k \end{aligned} \tag{4}$$

Similarly, $\phi^*_{k,\tau}$ is defined implicitly by the valuation of the corporate liability:

$$\begin{aligned} W^*_k &= \int_0^{B^*_k} v f_{Q^*}(v)dv + B^*_k \int_{B^*_k}^{\infty} f_{Q^*}(v)dv \\ &\equiv \phi^*_k B^*_k \end{aligned} \tag{5}$$

Combining (4) and (5) with (3), the arbitrage gain may be written as:

$$\begin{aligned} \mathbf{\Omega} &= B_k \left\{ \frac{1}{B^*_k} \int_0^{B^*_k} v f_{Q^*}(v)dv + \int_{B^*_k}^{\infty} f_{Q^*}(v)dv \right\} \\ &- B_k \left\{ \frac{1}{B_k} \int_0^{B_k} v f_Q(v)dv + \int_{B_k}^{\infty} f_Q(v)dv \right\} \end{aligned} \tag{6}$$

where B^*_k is given by equation (1) under a default probability rating system, and by equation (2) under a default probability rating system. Sufficient conditions for the arbitrage gain to be positive or negative are given in the following Lemma:

Lemma 1 *Default Probability*

(a) *The arbitrage gain, $\mathbf{\Omega}$, will be positive if P first order stochastically dominates P^* ($P \geq^{FSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the arbitrage gain will be negative if $P^* \geq^{FSD} P$ and $Q \geq^{SSD} Q^*$.*

(b) *Moreover if two SPVs have the same risk-neutral distribution Q and their physical distribution P_1 and P_2 are such that $P_2 \geq^{FSD} P_1 \geq^{FSD} P^*$ and $Q^* \geq^{SSD} Q$ then the arbitrage gain from issuing a structured bond with a given rating k will be greater for SPV_2 than for SPV_1.*

Proof: See Appendix

Lemma 2 *Expected Default Loss Rating System*

(a) *The arbitrage gain, $\mathbf{\Omega}$, will be positive if P second order stochastically dominates P^* ($P \geq^{SSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the arbitrage gain will be negative if $P^* \geq^{SSD} P$ and $Q \geq^{SSD} Q^*$.*

(b) *Moreover if two SPVs have the same risk-neutral distribution Q and their physical distribution P_1 and P_2 are such that $P_2 \geq^{SSD} P_1 \geq^{SSD} P^*$ and $Q^* \geq^{SSD} Q$ then the arbitrage gain from issuing a structured bond with a given rating k will be greater for SPV_2 than for SPV_1.*

Proof: See Appendix

As a direct application of part (a) of the Lemma 1, consider the situation in which either the single period CAPM or its continuous time version holds, and V and V^* have the same total risk. The risk neutral measures will then be identical: $Q \equiv Q^*$. P will first order stochastically dominate P^* whenever the SPV has a higher beta coefficient than the corporate issuer because this will imply a higher mean return. Part (b) of the lemma implies that, for a given total risk and bond rating, the arbitrage gain will be monotonically increasing in the beta of the SPV.

4.3 Multiple Tranches

Lemma 1 characterizes conditions under which the arbitrage gain from a simple securitization with a single tranche is positive. However, most securitizations involve multiple tranches.[16] In this section we consider when the arbitrage gain can be increased by issuing additional tranches. To analyze the gains from introducing multiply tranched securities, consider the gain from replacing a single tranche with face value B_k and rating k with two tranches. Denote the face value of the senior tranche by B_{1,k_1} and its rating by k_1, and denote the face value of the junior tranche by $B_{2,k_2} \equiv B_k - B_{1,k_1}$ and its rating by k_2.[17]

Under a *default probability* rating system, the default probability of the single tranche, Π_k, is equal to the default probability of the junior tranche of the dual tranche financing, since in both cases the SPV defaults when its terminal value, V, is less than $B_k = B_{1,k_1} + B_{2,k_2}$. As a result, the junior tranche sells at the same (corporate bond) yield as the single tranche: $\phi^*_{k_2} = \phi^*_k$. On the other hand, the senior tranche has a lower default probability than the single tranche issue so that it sells at a lower yield: $\phi^*_{k_1} > \phi^*_k$, and the gain from tranching is $(\phi^*_{k_1} - \phi^*_k)B_{1,k_1}$. It is straightforward to extend this argument to additional tranches as stated in the following lemma:

Lemma 3 *Default Probability Rating System*
Under a default probability rating system it is optimal to subdivide a given tranche into a junior and a senior tranche with different ratings.

The Lemma implies that is is optimal to have as many tranches as there are different rating classes.

Lemma 4 *Expected Default Loss Rating System*
Under an expected default loss rating system, if a given tranche is profitable, then it is optimal to subdivide the tranche into a junior and a senior tranche with different ratings, whenever the pricing kernel for the corporate issuer, $m^*(v)$, is a decreasing function of the underlying asset value.

Proof: See Appendix

Lemmas 3 and 4 are consistent with the findings of Cuchra and Jenkinson (2005) that the number of tranches in European securitisations has displayed a secular tendency to increase which they attribute to the growing sophistication of investors in these markets.

---

[16] Cuchra ans Jenkinson (2005) report that in 2003 the average number of tranches in European securitization was 3.93 tranches and in US securitization 5.58.

[17] Note that in our notation, B_{j,k_j}, j denotes the seniority of the tranche issued and k_j denotes its rating. Note that neither the payoff nor the rating of a given tranche depend on the existence or characteristics of more junior tranches.

5 Gains in the CAPM Framework

In order to quantify the gains from securitization when yields are set according to bond ratings we assume that asset returns satisfy the CAPM. In particular, we assume that the values of the underlying assets, i.e. the value of the portfolio held by the SPV (V), and the value of the assets of the firm on which the corporate bond is written (V^*), follow geometric Brownian motions:

$$dV = \mu V dt + \sigma V dz, \quad dV^* = \mu^* V^* dt + \sigma^* V^* dz^* \tag{7}$$

where

$$\mu = r_f + \beta(r_m - r_f), \quad \mu^* = r_f + \beta^*(r_m - r_f) \tag{8}$$

r_f denotes the risk-free rate, $(r_m - r_f)$ the excess market return, and β and β^* is the standard beta coefficient. It is of course a simplification to assume that asset values follow a geometric Brownian motion. However, our concern is not with the exact characteristics of the marginal distribution of the underlying portfolio value, but with its joint distribution with the pricing kernel and we maintain this assumption in the interests of analytical tractability.

Then the price dynamics of the corporate bond are fully described by the parameter set $\{\beta^*, \sigma^*\}$ and the portfolio of the Special Purpose Vehicle issuing the CDO is completely characterized by the parameter set $\{\beta, \sigma\}$.

5.1 Computation of Arbitrage Gain for a Simple Securitization

We assume that the issuer has determined the desired rating, k, of a single security tranche to be issued.

(i) Determination of B_k and B_k^*

When ratings are based on *default probabilities*, the face value of a tranche with rating k, B_k, can be determined from the expression for Π_k, the probability that the assets of the SPV are less than B_k at maturity:[18]

$$\Pi_k = \mathcal{N}\left(-\frac{ln(V/B_k) + (\mu - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \tag{9}$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution. Solving for B_k, we have:

$$B_k \equiv \frac{V}{exp\{-\mathcal{N}^{-1}[\Pi_k]\sigma\sqrt{\tau} - (\mu - 0.5\sigma^2)\tau\}} \tag{10}$$

Similarly B_k^* is determined by expression (10) with (V, μ, σ) replaced by (V^*, μ^*, σ^*).

When ratings are based on *expected default losses* we can determine the face value of a single tranche with rating k, B_k, by solving the following equation:

$$\Lambda_k = \frac{\mathcal{L}_k}{B_k} \tag{11}$$

[18] For convenience we again drop the maturity subscript τ although Π_k as well as B_k depend on the time to maturity.

where Λ_k denotes the expected loss rate of a claim with rating k (and maturity τ). The expected default loss, $\mathcal{L}_k$, is given by

$$\mathcal{L}_k = B_k \mathcal{N}(-d_2^P) - V e^{\mu\tau} \mathcal{N}(-d_1^P) \tag{12}$$

with

$$d_1^P = \frac{ln(V/B_k) + (\mu + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{13}$$

$$d_2^P = d_1^P - \sigma\sqrt{\tau} = \frac{ln(V/B_k) + (\mu - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{14}$$

Equation (12) uses the fact that the expected loss on a bond with nominal value B_k is the same as a put on the asset value, V, with strike B_k. B_k is implicitly determined by equations (12) to (14).[19] Similarly, the face value of the corporate bond with rating k, B_k^*, is determined by by equations (12) to (14) with (V, μ, σ) replaced by (V^*, μ^*, σ^*).

(ii) Determination of Market Values and Tranche Sales Price

The market value of the corporate bond, $W_k^*(V^*)$, is given by the *Merton* (1974) formula:

$$W_k^*(V^*) = B_k^* e^{-r_f\tau} \mathcal{N}(d_2^{Q^*}) + V^* \mathcal{N}(-d_1^{Q^*}) \tag{15}$$

where $d_1^{Q^*}$ and $d_2^{Q^*}$ are defined as in equations (13) and (14) substituting r_f for μ^* and (V^*, σ^*) for (V, σ). The market value of a single tranche, , $W_k(V)$, is determined in an analogous fashion.

Using the *Pricing Assumption*, that tranches and corporate bonds with the same ratings have identical yields, the sales price of the tranche is given by

$$S_k = \phi_k^* B_k = \frac{W_k^*(V^*)}{B_k^*} B_k \tag{16}$$

and the arbitrage gain equals

$$\mathbf{\Omega} = S_k - W_k(V) \tag{17}$$

5.2 Computation of the Arbitrage Gain for a Multiple Tranche Securitization

Since the corporate bond is assumed to be a senior security B_k^* is determined as in the single tranche case.

(i) Determination of B_k

Under a *probability of default* rating system, B_k can be interpreted as the sum over the nominal values of all tranches from the highest rating category to rating k. Given B_k and a desired tranche structure with N rated tranches one can

[19] Note that B_k enters also d_1^P and d_2^P such that one cannot directly solve for B_k.

determine the individual tranche sizes by taking differences. In particular, the size of tranche n with rating k_n is given by

$$B_{n,k_n} = B_{k_n} - B_{k_{n-1}} \tag{18}$$

where k_{n-1} denotes rating of the next better tranche. The size of the super-senior tranche, B_{1,k_1}, equals B_{k_1}.

Under an *expected default loss* rating system, the expected loss on the nth tranche with face value B_{n,k_n}, $\mathcal{L}_{n,k_n}$, is given by

$$\mathcal{L}_{n,k_n} = \mathcal{L}_{k_n} - \mathcal{L}_{k_{n-1}}$$

with $\mathcal{L}_{k_n}$ and $\mathcal{L}_{k_{n-1}}$ as defined in (12). Hence the expected loss rate on the nth tranche is given by

$$\Lambda_{k_n} = \frac{\mathcal{L}_{n,k_n}}{B_{n,k_n}} = \frac{\mathcal{L}_{k_n} - \mathcal{L}_{k_{n-1}}}{B_{k_n} - B_{k_{n-1}}} \tag{19}$$

For the highest rated tranche

$$\Lambda_{k_1} = \frac{\mathcal{L}_{k_1}}{B_{1,k_1}} = \frac{\mathcal{L}_{k_1}}{B_{k_1}} \tag{20}$$

which corresponds to equation (11). Given $\Lambda_{k_1}, \dots, \Lambda_{k_N}$ the implicit equations for B_{i,k_i}, (19) and (20), may be solved recursively starting with the highest rated tranche.

(ii) Determination of Market Values and Tranche Sales Price

The market value of the corporate bond with rating k, $W_k^*(V^*)$, is given as before by equation (15).

The market value of the nth tranche with nominal value B_{n,k_n} is given by the difference of the values of single tranches with face values B_{k_n} and $B_{k_{n-1}}$:

$$W_{n,k_n}(V) = W_{k_n}(V) - W_{k_{n-1}}(V)$$

with $W_{k_i}(V)$ and $W_{k_{i-1}}(V)$ as determined in the single tranche case.

Using the *Pricing Assumption* the sales price of the ith tranche, S_{i,k_i}, is given by

$$S_{i,k_i} = \phi_{k_i}^* B_{i,k_i} = \frac{W_{k_i}^*(V^*)}{B_{k_i}^*} B_{i,k_i} \tag{21}$$

and the arbitrage gain on the ith tranche is

$$\mathbf{\Omega}_i = S_{i,k_i} - W_{i,k_i}(V) \tag{22}$$

and the total profit is $\mathbf{\Omega} = \sum_i \mathbf{\Omega}_i$.

5.3 Quantitative Estimates of the Gains

In this section we present estimates of the gains to securitisation assuming a risk-free interest rate of 3.5 percent and a market risk premium of 8.5 percent.

Tables 4 and 5 illustrate the pricing of structured bonds with maturity of five years under default probability and expected default loss rating systems respectively, when the asset betas of both the SPV and the corporation underlying the bond ratings is 0.7. The total risk of the corporation is assumed to be 18% p.a., that of the SPV is 12%.

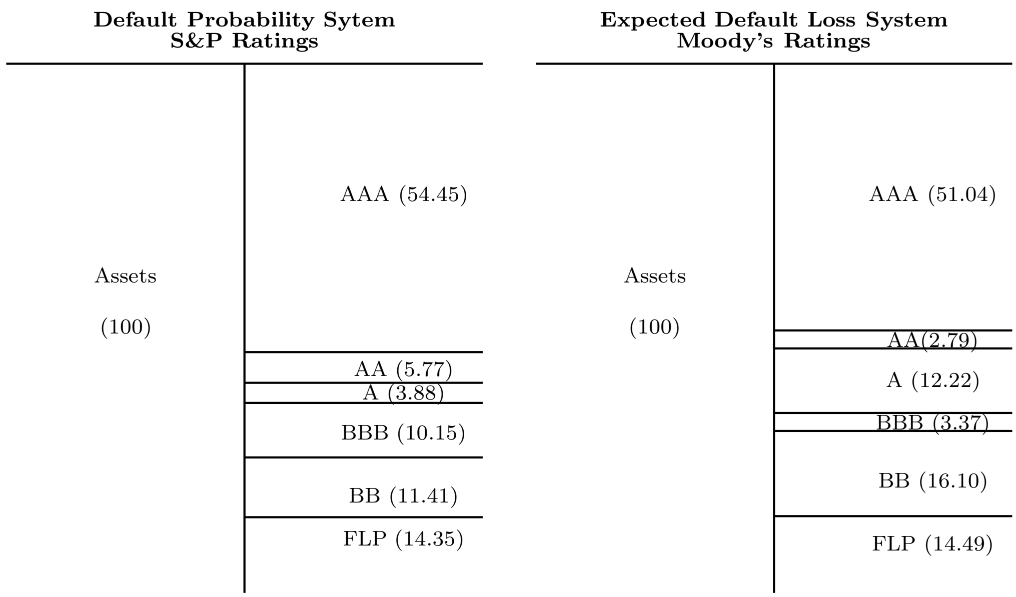
Panel A of Table 4 shows the pricing of corporate bonds and the determination of ϕ_k^* under a default probability rating system using the Standard& Poor's default frequencies. For each of the five bonds the default probability is taken from Table 1 and the face value of the bonds, B_k^*, and the market values, W_k^*, is calculated from equations (10) and (15). The expected default loss rate (which is not used in further calculations in this table) is calculated from equation (11). Column (4) of Panel B reports the face value, B_{k_i}, of an untranched bond issued by the SPV with probability of default Π_{k_i} taken from Table 1[20] calculated from equation (15). The face values of the untranched bond in the SPV exceed that of the corresponding corporate bonds, because the total risk of the SPV is less. The face values of the tranches is obtained by taking differences of the B_{k_i}. The market values of the untranched bonds, W_{k_i}, are determined by the Merton formula (15) and the market value of the tranches are obtained as first differences. The yields to maturity are continuously compounded. The sales price of tranche i, S_{i,k_i}, is obtained by multiplying the face value B_{i,k_i} by $\phi_{k_i}^*$. Finally, the gain is the difference between the sales price and the market value of each tranche. The unrated first loss piece (FLP) is assumed to be sold at its market value.[21] Comparing the equilibrium yield to maturity for securities with the same rating issued by the SPV and the corporation we note, that the equilibrium yields on the junior tranches of the securitisation significantly exceed those of the corresponding corporate bond. The equilibrium yield on the BB tranche is 10.95% as compared with 4.81% for the BB corporate bond. Computing the sales price it is assumed that the tranche yield are the same so that the sales price of the BB tranche of 15.51 substantially exceeds the equilibrium value 11.41. The gains on the higher rated tranches are proportionally smaller and the total gain from securitisation is 5.71. In this example, the gains arise primarily from the junior tranches. This contrasts with the suggestion of Coval it et al. (2007)who suggest that 'highly rated tranches should trade at significantly higher yield spreads than single name bonds with identical credit ratings.' Panel A shows that the equilibrium yield on the AAA corporate is 3.52%, while Panel shows that the equilibrium yield on the AAA tranche is 3.53%. Thus the yield difference on this tranche is only 1 basis point. In contrast the spread between the equilibrium yields on the BB tranche and the BB corporate bond is 6.14%.

Panel A of Table 5 calculates for each of the five tranches the face values

[20]This is conservative. If the probabilities were taken from Table 2, the arbitrage gains would be slightly higher.

[21]In practice the FLP is usually retained by the issuer, so that no gain is realized.

Figure 1: Equilibrium market value capital structures of an SPV under two different rating systems.



of the corporate bonds using Moody's default loss rates Λ_{k_i}. The equilibrium values, W_{k_i}, and price ratios, $\phi^*_{k_i}$, are derived as in Table 4. The face values of the untranched bonds in Panel B are now calculated to ensure that the default loss rate for each tranche is equal to Λ_{k_i}.[22] The remaining columns of Panel B are calculated in the same way as for Table 4. As under the default probability rating system, the arbitrage gain is concentrated in the junior tranche. It is not surprising, that the mispricing gain of 3.69 under the expected default loss rating system is smaller than the gain of 5.71 under the default probability system, which takes no account of the size of losses, when they occur.

Figure 1 illustrates the equilibrium market value capital structures of an SPV for the examples presented in Tables 4 and 5. Despite the conceptual differences between the Moody's and S&P rating systems the structures implied by these two systems are fairly similar. The senior tranche is 54.5% of the asset value under the S&P system and 51.0% under the Moody's system and in both cases the first loss piece is approximately 14%. The most significant differences arise in the allocations between AA and A and between BBB and BB tranches.

Table 6 reports arbitrage gains from securitisations with five and six tranches under a *default probability rating system*: The most junior of the five tranches has a S&P BB rating and the most junior of the six tranches has a B rating. The table shows for different assumptions about the collateral risk (β, σ) the total amount of debt that is issued and the arbitrage gain for both the five and six tranche securitisations $(\Omega^M_{BB}, \Omega^M_B)$ and the arbitrage gains from issuing just a single tranche with the same total market value $(\Omega^S_{BB}, \Omega^S_B)$. The difference

[22] See equations (19) and (20).

between $\Omega^M_\bullet$ and $\Omega^S_\bullet$ is the additional gain from issuing multiple tranches.

The total amount of debt that is issued is greater in the six tranche securitisation and is decreasing in the total risk of the collateral, σ. The results are consistent with the implication of Lemma 1 that a single tranche securitisation will be unprofitable in the cases marked by x and profitable in the cases marked by ✓. However most cases shown are not covered by the Lemma. Given the risk characteristics of the assets underlying the corporate bond rating (β^*, σ^*) the arbitrage gain from a single tranche securitisation is increasing in the systematic risk of the collateral, β, and generally decreasing in the total risk. Comparing Ω^S_{BB} and Ω^S_B for given collateral risk, the arbitrage gain from issuing a larger amount of debt in a single tranche is seen to be higher when $\beta > \beta^*$, and lower when $\beta^* > \beta$.

Consistent with Lemma 3, the arbitrage gain from replacing the single tranche with multiple tranches is always positive. The gain from the six tranche securitisation always exceeds that of the five tranche securitisation. The gain from multiple tranching is increasing in the systematic risk of the collateral, β, and decreasing in the total risk, σ. The potential gains are economically significant. From a five tranche securitisation, gains in the order of 5 to 8 percent of the collateral value are attainable in many cases and under a six tranche securitisation the gains rise to 12 to 18 percent.

Table 7 reports arbitrage gains from securitisations with five and six tranches under a *expected default loss rating system*: Note first that the total debt under the five tranche securitisation in which the junior tranche has a Moody's BB rating is almost identical to that under the default probability rating system when the junior tranche has a S&P BB rating. However for the six tranche securitisation Moody's ratings imply debt levels, which are five to eight percentage points lower. Single tranche securitisations are always profitable when the systematic risk of the collateral, β, exceeds that of the assets underlying the corporate bond, β^*, or when $\beta = \beta^*$ and $\sigma < \sigma^*$. When $\beta > \beta^*$ and $\sigma \leq \sigma^*$ the arbitrage gains from issuing a single tranche range from 0.79 to 5.78 percentage points when the market value of the debt corresponds to the total debt of a five tranche securitisation and from 1.01 to 7.12 percentage points in the six tranche case. Multiple tranching raises the arbitrage gain by 1 to 1.25 percentage points in the five tranche case and by 1.43 to 1.68 points in the six tranche case which is consistent with Lemma 4. Just as under a default probability rating system the arbitrage gains are increasing in systematic risk of the collateral and decreasing in the total risk, σ.

For the single tranche securitisations the arbitrage gain using Moody's ratings tends to be lower than those using S&P ratings for high collateral systematic and total risk. However, in all cases the gains for the multi-tranche securitisations are much higher for the S&P ratings than for the Moody's ratings.

6 Conclusion

In this paper we have analyzed the gains to asset securitisation in a market in which structured bonds can be sold to investors at prices/yields that reflect only their credit rating and credit ratings reflect either default probabilities or expected default losses. The former corresponds to the credit ratings of Standard and Poor's and the latter to the ratings of Moody's. For both rating systems we find general conditions under which single and multiple tranche securitisations will yield an arbitrage gain. The conditions depend on the risk characteristics of the collateral relative to those of the typical firm for which the bond ratings apply.

The analysis is then specialized to a market in which the continuous time CAPM holds and the Merton (1974) model is used to value both corporate bonds and securitisation tranches. We show that the arbitrage gains under both rating systems are highest when the systematic risk of the collateral is high and the total risk is low relative to the typical firm. In all cases we find significant additional gains to multi-tranching, which is consistent with the fact that there were 5.58 tranches in the average securitisation in the US in 2003.[23] The gains from issuing six tranches are higher than these from issuing five tranches. Finally we find that the arbitrage gains from multiple tranches are significantly higher when the securities are values using S&P ratings than using Moody's ratings.

Our analysis highlights the limitations of current credit rating systems which reflect characteristics of the *total* risk of fixed income securities, neglecting portfolio considerations. If ratings are to be used for valuation then it is important that they reflect the systematic risk of the securities.

A Appendix

A.1 Proof of Lemma 1

(a) If $P \geq^{FSD} P^*$, the first order stochastic dominance ranking of the physical distributions implies that under a default probability rating system or an, $B_k \geq B_k^*$. Then note that (6) can be written as:

$$\begin{aligned} \mathbf{\Omega} &= \frac{B_k}{B_k^*} E_{Q^*}\{min[B_k^*, V]\} - E_Q\{min[B_k, V]\} \qquad (23) \\ &= E_{Q^*}\left\{min[B_k, \frac{B_k}{B_k^*}V]\right\} - E_Q\{min[B_k, V]\} \\ &\geq E_{Q^*}\{min[B_k, V]\} - E_Q\{min[B_k, V]\} \qquad (24) \end{aligned}$$

$\mathbf{\Omega}$ is positive if $Q^* \geq^{SSD} Q$.

For the converse argument note that $P^* \geq^{FSD} P$ implies $B_k < B_k^*$.

(b) Note that if $P_2 \geq^{FSD} P_1$ the face value of the k-rated bond issued by the second SPV, B_k^2, is greater than the face value of bond issued by the first

[23]See Cuchra and Jenkinson (2005).

SPV, B_k^1. This implies that Ω^2 is greater than Ω^1 since expression (23) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.2 Proof of Lemma 2

(a) If $P \geq^{SSD} P^*$, the second order stochastic dominance ranking of the physical distributions implies that under an expected default loss rating system $B_k \geq B_k^*$. The rest of the proof follows from the proof of Lemma 1.

(b) If $P_2 \geq^{SSD} P_1$ the face value of the k-rated bond issued by the second SPV, B_k^2, is greater than the face value of bond issued by the first SPV, B_k^1. This implies that Ω^2 is greater than Ω^1 since expression (23) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.3 Proof of Lemma 4

$$\Delta\Omega = {\phi^*}_{k_1} B_{k_1} + {\phi^*}_{k_2} B_{k_2} - {\phi^*}_k B_k \tag{25}$$

Now

$$\phi^*_{k_1} \equiv \frac{E_{Q^*} min[B^*_{k_1}, V]}{B^*_{k_1}},\ \phi^*_{k_2} \equiv \frac{E_{Q^*} min[B^*_{k_2}, V]}{B^*_{k_2}},\ \phi^*_k \equiv \frac{E_{Q^*} min[B^*_k, V]}{B^*_k} \tag{26}$$

Therefore substituting from equations (26) in (25) and noting that $B_k = B_{1,k_1} + B_{2,k_2}$, we have:

$$\begin{aligned} \Delta\Omega &= \frac{B_{1,k_1}}{B^*_{k_1}} E_{Q^*} min[B^*_{k_1}, V] + \frac{B_{2,k_2}}{B^*_{k_2}} E_{Q^*} min[B^*_{k_2}, V] \\ &- \frac{B_{1,k_1} + B_{2,k_2}}{B^*_k} E_{Q^*} min[B^*_k, V] \end{aligned} \tag{27}$$

Now, under an expected default loss rating system, the SPV bonds have the same expected payoff per unit of face value as do the correspondingly rated corporate bonds, so that:

- for the untranched issue:

$$\frac{E_P min[B_k, V]}{B_k} = \frac{E_{P^*} min[B^*_k, V]}{B^*_k} \tag{28}$$

- for the senior tranche:

$$\frac{E_P min[B_{1,k_1}, V]}{B_{1,k_1}} = \frac{E_{P^*} min[B^*_{k_1}, V]}{B^*_{k_1}} \tag{29}$$

- for the junior tranche:

$$\frac{E_P \{min[B_k, V] - min[B_{1,k_1}, V]\}}{B_{2,k_2}} = \frac{E_{P^*} min[B^*_{k_2}, V]}{B^*_{k_2}} \tag{30}$$

Then substituting for B_k^*, $B_{k_1}^*$, and $B_{k_2}^*$ from equations (28)-(30) in (28):

$$\begin{aligned}\Delta\Omega &= \left\{\frac{E_{Q^*}min[B_{k_1}^*,V]}{E_{P^*}min[B_{k_1}^*,V]} - \frac{E_{Q^*}min[B_{k_2}^*,V]}{E_{P^*}min[B_{k_2}^*,V]}\right\} E_P min[B_{1,k_1},V] \\ &+ \left\{\frac{E_{Q^*}min[B_{k_2}^*,V]}{E_{P^*}min[B_{k_2}^*,V]} - \frac{E_{Q^*}min[B_k^*,V]}{E_{P^*}min[B_k^*,V]}\right\} E_P min[B_k,V] \qquad (31)\end{aligned}$$

Define the bond payoffs, $\pi_1^*(v) = min[B_{k_1}^*, v]$, $\pi_2^*(v) = min[B_{k_2}^*, v]$, $\pi^*(v) = min[B_k^*, v]$, $\pi_1(v) = min[B_{1,k_1}, v]$, $\pi_2(v) = min[B_{2,k_2}, v]$ and recall that $E_{Q^*}[v] = E_{P^*}[m^*(v)v]$. Then the incremental profit from the second tranche is

$$\begin{aligned}\Delta\Omega &= \left\{\frac{E_{P^*}[m^*\pi_1^*]}{E_{P^*}[\pi_1^*]} - \frac{E_{P^*}[m^*\pi_2^*]}{E_{P^*}[\pi_2^*]}\right\} E_P[\pi_1] \\ &+ \left\{\frac{E_{P^*}[m^*\pi_2^*]}{E_{P^*}[\pi_2^*]} - \frac{E_{P^*}[m^*\pi^*]}{E_{P^*}[\pi^*]}\right\} E_P[\pi_1 + \pi_2] \\ &= (E_P[\pi_1] + E_P[\pi_2])E_{P^*}[m^*(v)w(v)] \qquad (32)\end{aligned}$$

where

$$w_x(v) = x\left(\frac{\pi_1^*(v)}{E_{P^*}[\pi_1^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]}\right) + (1-x)\left(\frac{\pi_2^*(v)}{E_{P^*}[\pi_2^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]}\right) \qquad (33)$$

and $x = E_P[\pi_1(v)]/(E_P[\pi_1(v)] + E_P[\pi_2(v)])$. A second tranche will be profitable if there exists an x such that $E_{P^*}[m^*(v)w_x(v)] > 0$. $w_x(v)$ is a piecewise linear function with slopes given by:

$$\frac{dw_x(v)}{dv} = \begin{cases} x\left[\frac{1}{E_{P^*}[\pi_1^*]} - \frac{1}{E_{P^*}[\pi_2^*]}\right] + \left[\frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]}\right] & for\ v < B_{k_1}^* \quad (i) \\ (1-x)\frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]} & for\ B_{k_1}^* < v < B_k^* \quad (ii) \\ (1-x)\frac{1}{E_{P^*}[\pi_2^*]} & for\ B_k^* < v < B_{k_2}^* \quad (iii) \\ 0 & for\ v > B_{k_2}^* \quad (iv) \end{cases}$$

Note that the face value and therefore the expected payoff of a corporate bond is a decreasing function of its rating so that:

$$\frac{1}{E_{P^*}[\pi_1^*]} > \frac{1}{E_{P^*}[\pi^*]} > \frac{1}{E_{P^*}[\pi_2^*]}$$

Then for $0 \le x \le 1$ the slope dw_x/dv is negative in region (ii), positive in region (iii) and zero in region (iv). Note that $E_{P^*}[w_x(v)] = 0$. Consider $x = \hat{x}$ such that $w_{\hat{x}}(v) = 0$ in region (iv). Equation (33) implies that

$$\hat{x} = \frac{B_k^*/E_{P^*}[\pi^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}{B_{k_1}^*/E_{P^*}[\pi_1^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}$$

Since $E_{P^*}[w_x(v)] = 0$, the slope conditions in regions (ii) and (iii) imply that $w_{\hat{x}}(v) > 0$ in region (i), which is sufficient for $\mathbf{\Delta\Omega} \propto E_{P^*}[m^*(v)w_x(v)] > 0$ if $m^*(v)$ is a decreasing function.

References

[1] Black, F. and M, Scholes (1973): The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* Vol. 81 No. 3, 637-654.

[2] Bank of England (2007): *Financial Stability Report, October 2007*, Issue No. 22, London.

[3] Boot, A. and A. Thakor (1993): Security Design. *Journal of Finance* Vol. 48, 1349-1378.

[4] Brennan, M. and A. Kraus (1987): Efficient Financing under Asymmetric Information. *Journal of Finance* Vol. 42, 1225-1243.

[5] Coval, J.D., J. W. Jurek and E. Stafford (2007): Economic Catastrophe Bonds. *HBS Finance Working Paper* No. 07-102.

[6] Cuchra, Firla- M. (2005): Explaining Launch Spreads on Structured Bonds. *Discussion Paper*, University of Oxford.

[7] Cuchra, Firla- M., Jenkinson, T. (2005): Why Are Securitization Issues Tranched? *Working Paper* Department of Economics, Oxford University.

[8] DeMarzo, P. and D. Duffie (1999): A liquidity-based model of security design. *Econometrica* Vol. 67, 65-99.

[9] DeMarzo, P. (2005): The Pooling and Tranching of Securities: A Model of Informed Intermediation. *The Review of Financial Studies* Vol. 18, 1-35.

[10] Gaur, V.; Seshadri, S. and Marti Subrahmanyam, (2005): Intermediation and Value Creation in an Incomplete Market: Implications for Securitization. *Working Paper*, Leonard N. Stern School of Business, New York University.

[11] Fender, I.; Kiff, J. (2004): CDO rating methodology: Some thoughts on model risk and its implications. *BIS Working Papers* No 163.

[12] Franke, G., Th. Weber,. and M. Herrmann (2007): How does the market handle information asymmetries in securitizations? *Discussion Paper*, University of Konstanz.

[13] Grinblatt, M.,and F. A. Longstaff (2000): Financial Innovation and the Role of Derivative Securities: An Empirical Analysis of the Treasury STRIPS Program ,*Journal of Finance*55, 1415–1436

[14] Hein, J. (2007): Optimization of Credit Enhancements in Collateralized Loan Obligations - The Role of Loss Allocation and Reserve Account. *Discussion Paper*, University of Konstanz.

[15] Longstaff, F. and A. Rajan (2007): An Empirical Analysis of the Pricing of Collateralized Debt Obligations, *Journal of Finance*, forthcoming.

[16] Merton, R.C. (1974): On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance*, Vol. 29, 449-470.

[17] Modigliani, F. and M. H. Miller (1958): The Cost of Capital, Corporation Finance and the Theory of Investment. *American Economic Review*, Vol. 48 No. 3, 261-297.

[18] Moody's Investors Service (2005): *Special Comment: Default & Loss Rates of Structured Finance Securities: 1993-2004*, New York.

[19] Ross, St. (1989): Institutional Markets, Financial Marketing, and Financial Innovation,*Journal of Finance* Vol. 44 No. 3, 541-556 .

[20] Rubinstein, M. (1984): A Simple Formula for the Expected Rate of Return of an Option over a Finite Holding Period. *Journal of Finance*, Vol. 39 No. 5, 1503-1509.

[21] Securities Industry and Financial Markets Association (2008): *Research*, http://www.sifma.org/research/global-cdo.html.

[22] Standard & Poor's (2005): *CDO Evaluator Version 3.0: Technical Document*, London.

[23] Standard & Poor's (2007): *Structured Finance: Commentary*

[24] Stiglitz Joseph E. (1972) Some aspects of the pure theory of corporate finance: Bankruptcies and take-overs, *Bell Journal of Economics and Management Science*, Vol. 3 No. 2, 458-482.

Table 1:
Cumulative Default Frequencies for Corporate Issues (Standard & Poor's 2005).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.02	0.03	0.06	0.10	0.14
AA	0.01	0.04	0.09	0.14	0.22	0.31	0.42
A	0.02	0.08	0.17	0.30	0.46	0.66	0.89
BBB	0.29	0.68	1.16	1.71	2.32	2.98	3.67
BB	2.30	4.51	6.60	8.57	10.42	12.18	13.83
B	5.30	10.83	15.94	20.48	24.46	27.95	31.00

The table reports historical cumulative default frequencies (in percent) for the period 1981 to 2003 for 9,740 companies of which 1,386 defaulted.

Table 2:
Cumulative Default Frequencies for CDO tranches (Standard & Poor's 2005).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.03	0.07	0.12	0.19	0.29
AA	0.01	0.06	0.14	0.23	0.36	0.51	0.70
A	0.03	0.12	0.26	0.46	0.71	1.01	1.37
BBB	0.35	0.83	1.41	2.07	2.81	3.61	4.44
BB	2.53	4.95	7.23	9.38	11.40	13.31	15.11
B	5.82	11.75	17.15	21.92	26.09	29.73	32.90

The table reports cumulative default frequencies (in percent) based on "quantitative and qualitative considerations" (Standard & Poor's 2005, p. 10).

Table 3: Cumulative 'Idealized Loss Rates' according to Moody's (2005).

	1	2	3	4	5	6	7
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Aa	0.00	0.00	0.01	0.03	0.04	0.05	0.06
A	0.01	0.04	0.12	0.19	0.26	0.32	0.39
Baa	0.09	0.26	0.46	0.66	0.87	1.08	1.33
Ba	0.86	1.91	2.85	3.74	4.63	5.37	5.89
B	3.94	6.42	8.55	9.97	11.39	12.46	13.21

Table 4: Pricing SPV liabilities using a *Default Probability* rating system

Panel A: Corporate Bond Valuation by Rating Class

i	S&P Rating (k_i)	Probability of Default Π_{k_i}	(Exp. Loss) (Λ_{k_i})	Face Value $B^*_{k_i}$	Value $W^*_{k_i}$	Equilibrium Yield to Maturity	ϕ^*_k
1	AAA	0.061%	(0.006%)	40.24	33.76	3.52%	0.839
2	AA	0.219%	(0.023%)	46.98	39.35	3.55%	0.838
3	A	0.459%	(0.052%)	51.84	43.33	3.59%	0.836
4	BBB	2.323%	(0.307%)	66.37	54.75	3.85%	0.825
5	BB	10.424%	(1.711%)	89.16	70.09	4.81%	0.786

Panel B: Structured Bond Valuation and Sales Prices by Rating Class

i	S&P Rating (k_i)	Probability of Default Π_{k_i}	(Exp. Loss) (Λ_{k_i})	ϕ^*_k	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	AAA	0.061%	(0.006%)	0.839	64.97	64.97	54.46	54.46	3.53%	54.50	0.04
2	AA	0.219%	(0.023%)	0.838	72.03	7.07	60.22	5.77	4.06%	5.92	0.15
3	A	0.459%	(0.052%)	0.836	76.91	4.88	64.10	3.88	4.60%	4.08	0.20
4	BBB	2.323%	(0.307%)	0.825	90.68	13.78	74.25	10.15	6.11%	11.37	1.22
5	BB	10.424%	(1.711%)	0.786	110.41	19.72	85.65	11.41	10.95%	15.51	4.10
-	FLP						100.00	14.35		14.35	0.00
									Total:	105.71	5.71

Parameter Assumptions: $V^*(0) = V(0) = 100$, $\tau = 5$, $r_f = 3.5\%$, $r_m - r_f = 8.5\%$, $(\beta^*; \sigma^*) = (0.7; 0.18)$ and $(\beta; \sigma) = (0.7; 0.12)$

Table 5: Pricing SPV liabilities using a *Expected Default Loss* rating system

Panel A: Corporate Bond Valuation by Rating Class

i	Moody's Rating (k_i)	Exp. Loss Λ_{k_i}	(Probability) of Default (Π_{k_i})	Face Value $B^*_{k_i}$	Equilibrium Value $W^*_{k_i}$	Equilibrium Yield to Maturity	ϕ^*_k
1	Aaa	0.002%	(0.018%)	35.11	29.47	3.51%	0.839
2	Aa	0.037%	(0.339%)	49.75	41.62	3.57%	0.837
3	A	0.257%	(1.978%)	64.60	53.41	3.80%	0.827
4	Baa	0.869%	(5.824%)	78.64	63.48	4.28%	0.807
5	Ba	4.626%	(23.595%)	110.73	80.84	6.29%	0.730

Panel B: Structured Bond Valuation and Sales Prices by Rating Class

i	Moody's Rating (k_i)	Exp. Loss Λ_{k_i}	(Probability) (of Default) (Π_{k_i})	ϕ^*_k	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	Aaa	0.002%	(0.025%)	0.839	60.85	60.85	51.05	51.05	3.52%	51.07	0.025
2	Aa	0.037%	(0.052%)	0.837	64.20	3.35	53.82	2.78	3.74%	2.80	0.24
3	A	0.257%	(0.647%)	0.827	79.42	15.22	66.05	12.22	4.38%	12.58	0.36
4	Baa	0.869%	(1.125%)	0.807	83.88	4.47	69.41	3.37	5.65%	3.61	0.24
5	Ba	4.626%	(10.239%)	0.730	110.10	26.22	85.51	16.10	9.76%	19.14	3.05
-	FLP						100.00	14.49		14.49	0.00
									Total:	103.69	3.69

Parameter Assumptions: $V^*(0) = V(0) = 100$, $\tau = 5$, $r_f = 3.5\%$, $r_m - r_f = 8.5\%$, $(\beta^*;\sigma^*) = (0.7; 0.18)$ and $(\beta;\sigma) = (0.7; 0.12)$

Table 6:
Arbitrage Gains from Securitisation under a *Default Probability Rating System*

Collateral			Five Tranches			Six Tranches		
β	σ	Lemma 1 (a)	Total Debt	Ω^M_{BB}	Ω^S_{BB}	Total Debt	Ω^M_B	Ω^S_B
0.6	0.12	x	83.4	4.18	-0.19	90.8	9.41	-1.28
	0.14	x	78.1	3.44	-0.49	87.1	8.35	-1.46
	0.16	x	72.9	2.88	-0.63	83.3	7.51	-1.49
	0.18		67.9	2.45	-0.69	79.4	6.81	-1.42
	0.20		63.0	2.11	-0.69	75.5	6.22	-1.31
0.7	0.12		85.7	5.71	1.14	92.4	12.11	0.96
	0.14		80.4	4.65	0.56	88.9	10.68	0.44
	0.16		75.2	3.87	0.21	85.2	9.53	0.15
	0.18		70.1	3.27	0	81.4	8.59	0
	0.20		65.1	2.80	-0.12	77.4	7.80	-0.05
0.8	0.12		87.8	7.55	2.79	93.9	15.20	3.56
	0.14		82.7	6.11	1.84	90.6	13.33	2.64
	0.16		77.5	5.06	1.24	87.0	11.84	2.05
	0.18		72.3	4.25	0.84	83.2	10.62	1.66
	0.20	✓	67.3	3.62	0.57	79.4	9.59	1.40
0.9	0.14		84.8	7.85	3.39	92.1	16.34	5.19
	0.16		79.6	6.46	2.47	88.7	14.46	4.24
	0.18		74.5	5.40	1.84	85.0	12.92	3.56
	0.20	✓	69.4	4.58	1.34	81.2	11.62	3.07
1.0	0.16		81.7	8.10	3.95	90.2	17.40	6.73
	0.18		76.6	6.75	3.03	86.7	15.49	5.73
	0.20	✓	71.5	5.69	2.38	82.9	13.89	4.97
1.1	0.18		78.7	8.30	4.42	88.3	18.36	8.17
	0.20	✓	73.5	6.98	3.52	84.6	16.42	7.11
1.2	0.18		80.6	10.09	6.04	89.7	21.53	10.90
	0.20	✓	75.6	8.46	4.85	86.2	19.22	9.51

The characteristics of the firm on which the ratings are based are $\beta^* = 0.7$ and $\sigma^* = 0.18$. In addition $r_f = 3.5\%$ and $r_m - r_f = 8.5\%$. β and σ are the systematic and total risk parameters of the collateral underlying the securitisation. Lemma 1 (a) provides sufficient conditions for a gain (✓) or a loss (x) from a single tranche securitisation. The table shows results for a five tranche securitisation with ratings AAA, AA, A, BBB and BB and a six tranche securitisation with an additional B tranche. Total debt is the sum of the equilibrium market values of the issued tranches. Ω^M_{BB} (Ω^M_B) is the arbitrage gain from a five (six) tranche securitisation expressed as percent of the underlying collateral value. Ω^S_{BB} (Ω^S_B) is the arbitrage gain from a *single tranche* securitisation with the same total amount of debt as the corresponding multi-tranche securitisation.

Table 7:
Arbitrage Gains from Securitisation under an *Expected Loss Rating System*

Collateral			Five Tranches			Six Tranches		
β	σ	Lemma 2 (a)	Total Debt	Ω^M_{Ba}	Ω^S_{ba}	Total Debt	Ω^M_B	Ω^S_b
0.6	0.12		83.2	2.27	1.10	85.0	2.72	1.25
	0.14		77.7	1.38	0.29	80.2	1.77	0.33
	0.16		72.3	0.74	-0.26	75.4	1.04	-0.34
	0.18	x	67.0	0.27	-0.65	70.8	0.49	-0.84
	0.20	x	62.0	-0.07	-0.91	66.2	0.04	-1.21
0.7	0.12	✓	85.5	3.70	2.48	87.2	4.36	2.83
	0.14	✓	80.1	2.48	1.34	82.4	3.09	1.58
	0.16	✓	74.6	1.61	0.55	77.6	2.12	0.67
	0.18		69.3	0.96	0.00	73.0	1.38	0.00
	0.20		64.1	0.48	-0.39	68.3	0.79	-0.51
0.8	0.12	✓	87.6	5.43	4.15	89.2	6.33	4.74
	0.14	✓	82.3	3.81	2.63	84.6	4.68	3.11
	0.16	✓	76.9	2.65	1.56	79.4	3.42	1.85
	0.18	✓	71.5	1.79	0.79	75.1	2.45	1.01
	0.20		66.3	1.15	0.24	70.4	1.68	0.33
0.9	0.14	✓	84.4	5.41	4.16	86.6	6.55	4.91
	0.16	✓	79.1	3.90	2.77	81.9	4.95	3.36
	0.18	✓	73.7	2.79	1.75	77.2	3.71	2.20
	0.20		68.4	1.95	1.00	72.5	2.74	1.32
1.0	0.16	✓	81.2	5.39	4.19	83.9	6.72	5.07
	0.18	✓	75.9	3.97	2.89	79.2	5.17	3.60
	0.20		70.5	2.90	1.91	74.6	3.96	2.49
1.1	0.18	✓	77.9	5.36	4.20	81.2	6.86	5.23
	0.20		72.5	4.00	2.96	76.6	5.38	3.85
1.2	0.18	✓	80.0	6.95	5.78	83.1	8.80	7.12
	0.20		74.5	5.30	4.18	78.5	7.00	5.39

The characteristics of the firm on which the ratings are based are $\beta^* = 0.7$ and $\sigma^* = 0.18$. In addition $r_f = 3.5\%$ and $r_m - r_f = 8.5\%$. β and σ are the systematic and total risk parameters of the collateral underlying the securitisation. Lemma 2 (a) provides sufficient conditions for a gain (✓) or a loss (x) from a single tranche securitisation. The table shows results for a five tranche securitisation with ratings AAA, AA, A, BBB and BB and a six tranche securitisation with an additional B tranche. Total debt is the sum of the equilibrium market values of the issued tranches. Ω^M_{BB} (Ω^M_B) is the arbitrage gain from a five (six) tranche securitisation expressed as percent of the underlying collateral value. Ω^S_{ba} (Ω^S_b) is the arbitrage gain from a *single tranche* securitisation with the same total amount of debt as the corresponding multi-tranche securitisation. Note that unlike under the default probability rating system the rating of the single tranche is no longer Ba (B).