**Stat 316 Stochastic Processes on Graphs**

# Ising models on $\mathbb{Z}^d$

*Andrea Montanari*

*Lecture 5 - 10/8/2007*

This lecture lies a bit outside the line of our course, in that we shall consider a non-mean field setting. More precisely, we let $G = (V, E)$ be a square grid of side $\sqrt{N}$ and consider the Ising model on such a grid:

$$\mu(x) = \frac{1}{Z(\beta)} \exp\Big\{\beta \sum_{(ij)\in E} x_i x_j\Big\}, \tag{1}$$

and will prove the following result.

**Theorem 1.** *Let $X$ be a random configuration form the two-dimensional Ising model (1) and $\overline{X} = N^{-1}\sum_{i=1}^N X_i$ denote its magnetization.*

*Then there exist two inverse temperatures $\beta_1 > \beta_2 > 0$, such that the following happens. If $\beta > \beta_1$ ('low-temperature'), then there exist $\delta(\beta) > 0$ such that*

$$\mathbb{E}|\overline{X}| \geq \delta(\beta). \tag{2}$$

*If $\beta < \beta_2$ ('high-temperature'), then, for any $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}\{|\overline{X}| \leq \delta\} = 1. \tag{3}$$

Our proofs will be based on two classical argument as found in [Gri64] (low temperature) and [Fis67] (high temperature). While the above statement and our proofs refer to the two-dimensional case, the techniques are more general.

## 1 Low temperature: Peierls argument

The *dual lattice* of $G$ is the graph $G^* = (V^*, E^*)$ constructed as follows. Imagine to draw $G$ on a plane, using the standard embedding of $\mathbb{Z}^2$ in $\mathbb{R}^2$. Then, for each edge $(i, j)$ in the original lattice $G$, draw a perpendicular edge of length 1, centered at the midpoint of $(i, j)$. These are the edges of the $G^*$. Its vertices are the endpoints of such edges (only whenever two endpoints coincide, the vertices do coincide as well).

The number of vertices in the dual lattice is $|V^*| = (\sqrt{N} + 1)^2 - 4$.

A *Peierls contours* configuration is a set $\mathfrak{C} = \{C_1, \dots, C_n\}$ of directed paths $C_i$ in $G^*$ that satisfy the following conditions:

($i$) Each path is either closed or has on the boundary (degree-one vertices) of $G_*$.

($ii$) No two paths cross each other.

($iii$) Whenever two paths share a vertex, each of them bends to the right.

($iv$) Whenever path $C_i$ is 'directly inside' path $C_j$, the two paths have opposite orientations.

In the last point, by 'directly inside' we mean the following. If $C_i$ and $C_j$ are closed paths, then it must be that $C_i$ is inside $C_j$ but not inside any other path $C_k$ which in turn is inside $C_j$. If they are not closed, we first close them around the lattice and then require the same condition to be satisfied.

To a given Peierls contours configuration we associate the probability

$$\mu_*(\mathfrak{C}) = \frac{1}{Z^*(\beta)} \prod_{C\in\mathfrak{C}} e^{-2\beta|C|}. \tag{4}$$

**Lemma 2.** *There is a one-to-one correspondence between Ising model configurations and Peierls contours configurations. Further, under this correspondence, the Ising measure $\mu(x)$ is mapped to the Peierls one $\mu_*(\mathfrak{C})$.*

*Given an Ising configuration $x$, the corresponding Peierls contours separate the vertices with $x_i = +1$ from those with $x_i = -1$.*

**Proof** By figures. $\square$

**Proof** [Theorem 1, low temperature.] For the sake of simplicity we shall prove that $\mathbb{P}\{|\overline{X}| = 0\}$ is bounded as in the statement. The cases $\mathbb{P}\{|\overline{X}| = k\}$ with $|k| \leq N\delta$ can be treated analogously and the thesis (2) follows by union bound.

First notice that the number of Peierls contours $C$ with $|C| = l$ is upper bounded as

$$n(l) \leq |V^*| 3^l \leq N\, 3^{l+1}\,. \tag{5}$$

This is showed by first choosing a starting point (for which there are $|V^*|$ possibilities), and then a length-$l$ non-reversing walk in $\mathbb{Z}^2$.

The second remark is that, for any contour $C$

$$\mu_*(C \in \mathfrak{C}) \leq e^{-2\beta|C|}\,, \tag{6}$$

where $|C|$ denotes the length of $C$. This is proved by writing

$$\mu_*(C \in \mathfrak{C}) = \frac{\sum_{\mathfrak{C}\ni C} e^{-2\beta|\mathfrak{C}|}}{\sum_{\mathfrak{C}'} e^{-2\beta|\mathfrak{C}'|}}\,, \tag{7}$$

and noticing that, for each configuration $\mathfrak{C}$ in the numerator, there exists a configuration $\mathfrak{C}' = \mathfrak{C} \setminus C$ in the denominator. The bound follows by restricting the sum in the denominator to such terms and using $|\mathfrak{C}'| = |\mathfrak{C}| - |C|$.

Given a contours configuration $\mathfrak{C}$, any contour $C \in \mathfrak{C}$ divides $V$ in two subsets: those *inside* $C$ and those *outside* $C$. If $C$ is closed, the definition of 'inside' and 'outside' is obvious. If it is not closed, we will call 'inside' the smallest of the two subsets in which $C$ divides $V$ (an arbitrary convention can be chosen if they are equal).

We now divide the set of Ising configurations in two subsets

$\mathsf{P}(+)$ All the vertices $i \in V$ with $x_i = -1$ lie inside some contour.

$\mathsf{P}(-)$ All the vertices $i \in V$ with $x_i = +1$ lie inside some contour.

It is clear that $\mathsf{P}(+)$ and $\mathsf{P}(-)$ are in one-to one correspondence under the mapping $\{x_i\} \mapsto \{-x_i\}$. Further any configuration $x$ is either in $\mathsf{P}(+)$ or in $\mathsf{P}(-)$.

If we denote by $\mathbb{I}_{\mathsf{P}(\pm)}$ the indicator function on $\mathsf{P}(\pm)$, and by $N_-$ the number of vertices $i \in V$, with $x_i = -1$, we obtain

$$\mathbb{E}|\overline{X}| = \mathbb{E}\{|\overline{X}|\, \mathbb{I}_{\mathsf{P}(+)}\} + \mathbb{E}\{|\overline{X}|\, \mathbb{I}_{\mathsf{P}(-)}\} \geq \mathbb{E}\{\overline{X}\, \mathbb{I}_{\mathsf{P}(+)}\} - \mathbb{E}\{\overline{X}\, \mathbb{I}_{\mathsf{P}(-)}\} = \tag{8}$$

$$= 2\mathbb{E}\{\overline{X}\, \mathbb{I}_{\mathsf{P}(+)}\} = 1 - \frac{2}{N}\mathbb{E}\{N_-\, \mathbb{I}_{\mathsf{P}(+)}\}\,. \tag{9}$$

If $x \in \mathsf{P}(+)$, $N_-$ is upper bounded by the number of vertices that are inside at least one contour, which is in turn not larger than $\sum_{C\in\mathfrak{C}} |C_i|^2$. Therefore

$$\mathbb{E}|\overline{X}| \geq 1 - \frac{2}{N}\mathbb{E}\left\{\sum_{C\in\mathfrak{C}} |C|^2\right\} = 1 - \frac{2}{N}\sum_{C} |C|^2 \mu_*(C \in \mathfrak{C}) \geq$$

$$\geq 1 - \frac{2}{N}\sum_{\ell\geq 2} \ell^2 n(\ell) e^{-2\beta\ell} \geq 1 - 2\sum_{\ell \geq 2} \ell^2 (3e^{-2\beta})^\ell\,.$$

It is clear that the last expression is larger than, say, 1/2 for all $\beta$ large enough. $\square$

## 2 High-temperature expansion

As in the low-temperature case, the first step consists in proving an appropriate 'geometrical' representation of the sums involved in taking expectation with respect to the Ising model. To this purpose, given a subset $U \subseteq V$ of vertices, we let

$$Z_U(\beta) \equiv \sum_{x\in\{+1,-1\}^V} \prod_{i\in U} x_i \, \exp\Big\{\beta \sum_{(ij)\in E} x_i x_j\Big\}. \tag{10}$$

**Lemma 3.** *For $U \subseteq V$, let $\mathfrak{G}(U)$ denote the set of subgraphs of $G$ that have odd-degree in $U$ and even degree (eventually vanishing) elsewhere. Then (denoting by $E(F)$ the set of edges of the subgraph $F$ and by $|E(F)|$ its size)*

$$Z_U(\beta) = 2^{|V|}(\cosh\beta)^{|E|} \sum_{F\in\mathfrak{G}(U)} (\tanh\beta)^{|E(F)|}. \tag{11}$$

**Proof** Follows from the identity

$$\exp\Big\{\beta \sum_{(ij)\in E} x_i x_j\Big\} = (\cosh\beta)^{|E|} \prod_{(i,j)\in E} (1 + x_i x_j \, \tanh\beta) \tag{12}$$

expanding the product on the right hand side and summing over $x$. $\square$

**Lemma 4.** *For $i, j \in V$, let $d(i,j)$ denote their graph theoretical distance. There exists $\beta_2 > 0$ such that, for all $\beta \leq \beta_2$, there exists constants $A(\beta), \lambda(\beta) \geq 0$ with $\lambda(\beta) \leq 1$, such that*

$$\mathbb{E}\{X_i X_j\} \leq A(\beta)\, \lambda(\beta)^{d(i,j)}. \tag{13}$$

**Proof** By the previous Lemma, and writing $t = \tanh\beta < 1$, we have

$$\mathbb{E}\{X_i X_j\} = \frac{\sum_{F\in\mathfrak{G}(\{i,j\})} t^{|E(F)|}}{\sum_{F'\in\mathfrak{G}(\emptyset)} t^{|E(F')|}}. \tag{14}$$

Notice that, for each connected component of the subgraph $F$, the sum of the degrees of its vertices is even. As a consequence, any $F \in \mathfrak{G}(\{i,j\})$ includes a path, call it $F_{ij}$, connecting $i$ to $j$. Call the set of such paths $\mathfrak{P}(i,j)$. Further denote by $\mathfrak{G}(\emptyset; F_{ij})$ the set of subgraphs in $\mathfrak{G}(\emptyset)$ that have no edge in common with $F_{ij}$. Then we have

$$\mathbb{E}\{X_i X_j\} \leq \sum_{F_{ij}\in\mathfrak{P}(i,j)} t^{|E(F_{ij})|} \frac{\sum_{F'\in\mathfrak{G}(\emptyset;F_{ij})} t^{|E(F')|}}{\sum_{F'\in\mathfrak{G}(\emptyset)} t^{|E(F')|}} \leq \sum_{F_{ij}\in\mathfrak{P}(i,j)} t^{|E(F_{ij})|}. \tag{15}$$

The number of paths in $\mathfrak{P}(i,j)$ of length $l$ is upper bounded by $3^l$, and their minimal length $d(i,j)$. Therefore

$$\mathbb{E}\{X_i X_j\} \leq \sum_{l\geq d(i,j)} (3t)^l \leq (1-3t)^{-1}\,(3t)^{d(i,j)} \tag{16}$$

which proves the thesis. $\square$

**Proof** [Theorem 1, high temperature.] Notice that

$$\mathbb{E}\{\overline{X}^2\} = \frac{1}{N^2} \sum_{i,j\in V} \mathbb{E}\{X_i, X_j\} \le \frac{1}{N^2} \sum_{i,j\in V} A\,\lambda^{d(i,j)} \le \frac{A}{N} \sum_{j\in\mathbb{Z}^2} \lambda^{d(i,j)} \le \frac{A}{N} \sum_{d=0}^{\infty} 8d\,\lambda^d \tag{17}$$

Since the last sum converges for $\lambda < 1$, we have $\mathbb{E}\{\overline{X}^2\} \le C(\beta)/N$ for some constant $C(\beta)$ and all $\beta < \beta_2$, whence the thesis follows. □

# References


[Fis67] Michael E. Fisher. Critical Temperatures of Anisotropic Ising Lattices. II. General Upper Bounds. *Phys. Rev.*, 162:480–485, 1967.

[Gri64] Robert B. Griffiths. Peierls Proof of Spontaneous MAgnetization in a Two-Dimensional Ising Ferromagnet. *Phys. Rev.*, 136:A437–A439, 1964.