

Theorem Lemma[section] Corollary[section] Definition[section]
[theorem]Fact

Local Privacy and Statistical Minimax Rates

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University of California, Berkeley

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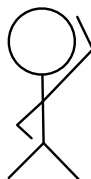
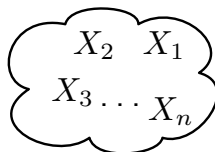
Goals for this talk

Bring together some classical concepts of decision theory
and newer concepts of privacy

Illustration of problem

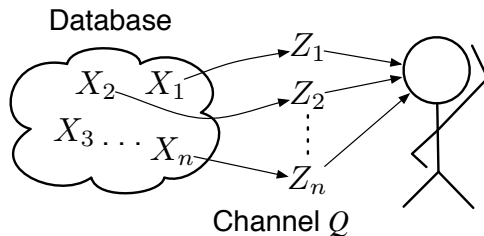
Illustration of problem

Database



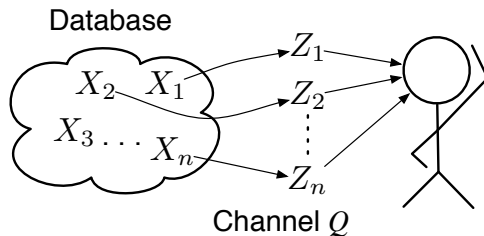
- ▶ Have data $X_1, \dots, X_n$

Illustration of problem



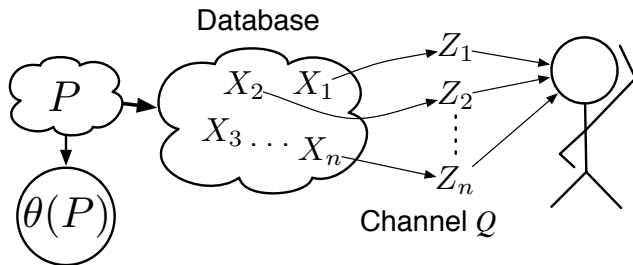
- ▶ Have data $X_1, \dots, X_n$
- ▶ **Private** views $Z_1, \dots, Z_n$ constructed from X_i

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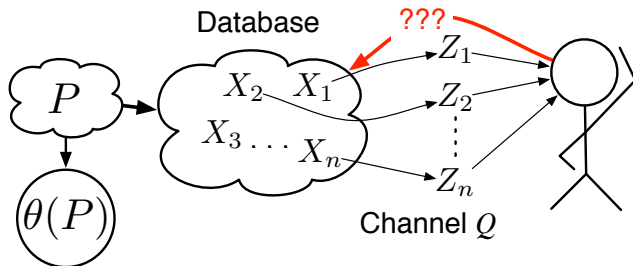
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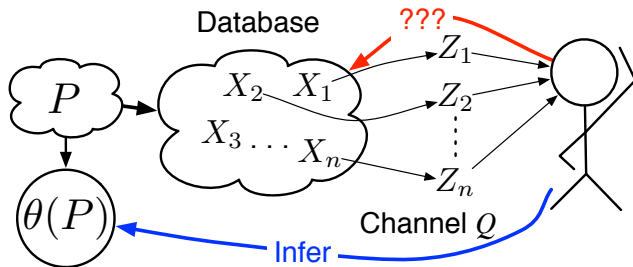
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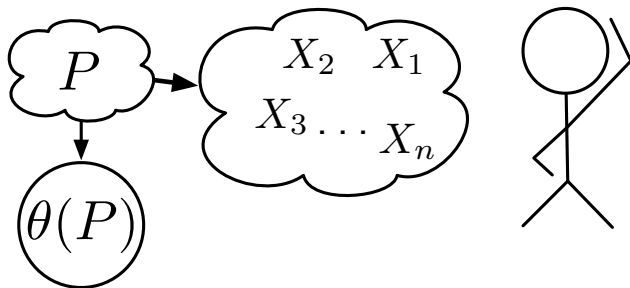
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- ▶ Goal: infer **population** parameter $\theta(P)$ based on $Z_1, \dots, Z_n$

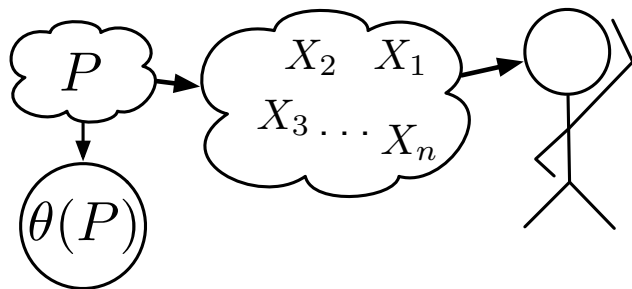
Primer on minimax rates of convergence and statistical inference

Illustration of classical problem



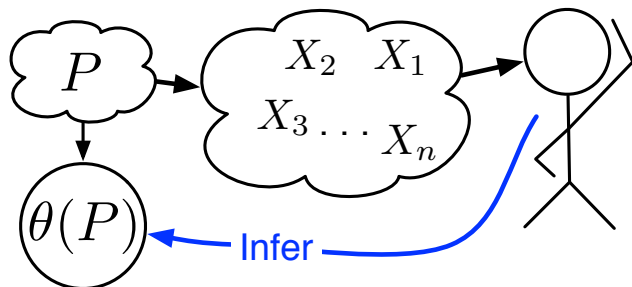
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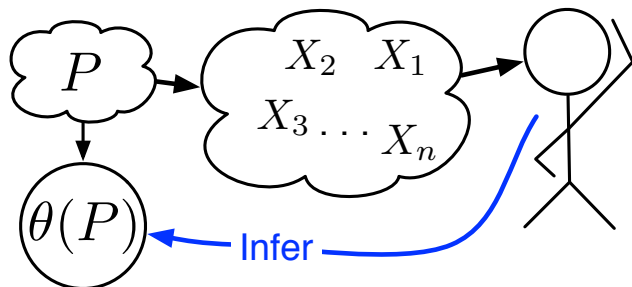
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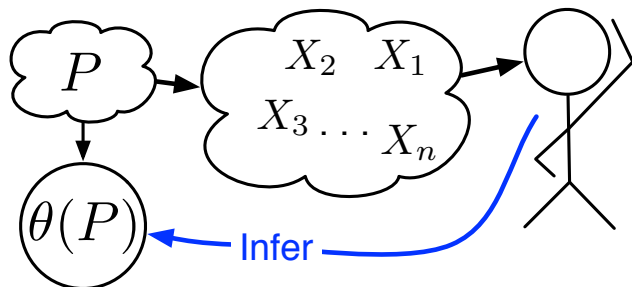
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Why? Care about making future predictions

Illustration of classical problem

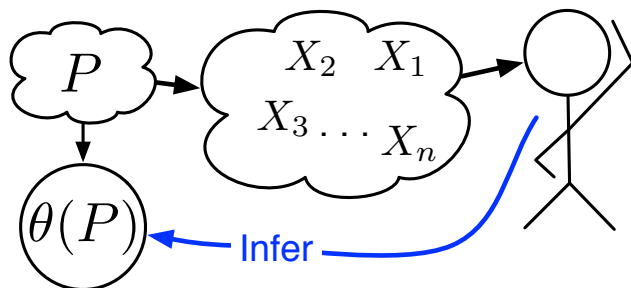


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- ▶ What is likelihood new resident of San Francisco needs food stamps

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Why? Care about making future predictions

- ▶ What is likelihood new resident of San Francisco needs food stamps
- ▶ Biological prediction, web advertising, search, ...

Minimax risk

Central object of study: Minimax risk

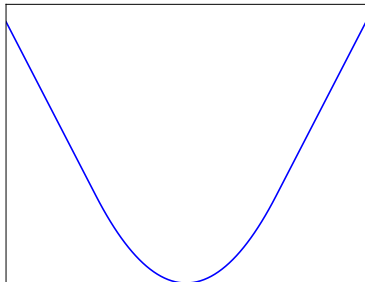
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- ▶ Parameter $\theta(P)$ of distribution P
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 - ▶ E.g. $\rho(\hat{\theta}, \theta) = \|\hat{\theta} - \theta\|_2^2$ or more esoteric/robust

$$\begin{aligned} & \rho_u(\hat{\theta}, \theta) \\ &= \begin{cases} \frac{1}{2} \|\hat{\theta} - \theta\|_2^2 & \text{if } \|\hat{\theta} - \theta\|_2 \leq u \\ u \|\hat{\theta} - \theta\|_2 - u^2/2 & \text{otherwise} \end{cases} \end{aligned}$$

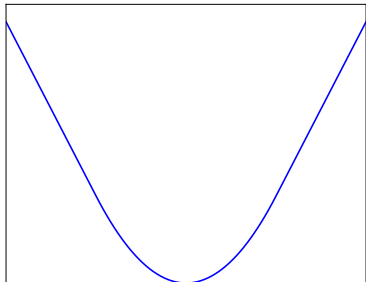


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- ▶ Family of distributions $\mathcal{P}$ that we study
 - ▶ E.g. P such that $\mathbb{E}_P[X^2] \leq 1$

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Look at expected loss

$$\mathbb{E}_P \left[\rho(\hat{\theta}(X_1, \dots, X_n), \theta(P)) \right]$$

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$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[\rho(\hat{\theta}(X_1, \dots, X_n), \theta(P)) \right]$$

- ▶ Worst case over distributions $\mathcal{P}$
- ▶ Best case over all estimators $\hat{\theta} : \mathcal{Z}^n \rightarrow \Theta$

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$$\mathfrak{M}_n(\theta(\mathcal{P}), \rho) := \underbrace{\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[\rho(\hat{\theta}(X_1, \dots, X_n), \theta(P)) \right]}_{\text{Classical minimax risk}}$$

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To study: rate of $\mathfrak{M}_n(\theta(\mathcal{P}), \rho) \rightarrow 0$ as n grows

Proving minimax bounds

This talk:

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- ▶ Upper bounds will be ad-hoc

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- ▶ Upper bounds will be ad-hoc
- ▶ Lower bounds will be information theoretic [Hasminskii 78, Birge 83, Ibragimov and Hasminskii 81, Yang and Barron 99, Yu97]

Proving minimax bounds

This talk:

- ▶ Upper bounds will be ad-hoc
- ▶ Lower bounds will be information theoretic [Hasminskii 78, Birge 83, Ibragimov and Hasminskii 81, Yang and Barron 99, Yu97]
- ▶ **NB:** Many known information-theoretic upper bounds [Barron, Birge, Kivinen, ...]

Proving minimax lower bounds

Step 1: Reduce from estimation to testing

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Step 2: Canonical testing problem

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Step 3: Classical bounds on error probabilities

Proving minimax lower bounds

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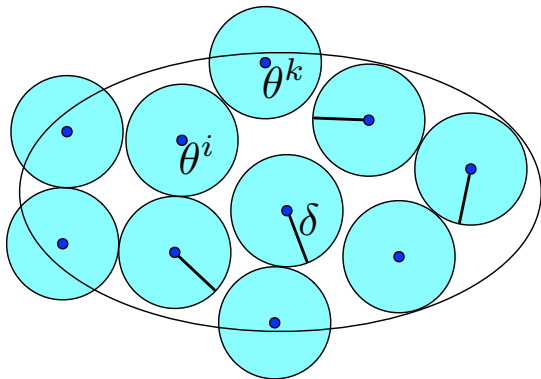
Step 3: Classical bounds on error probabilities

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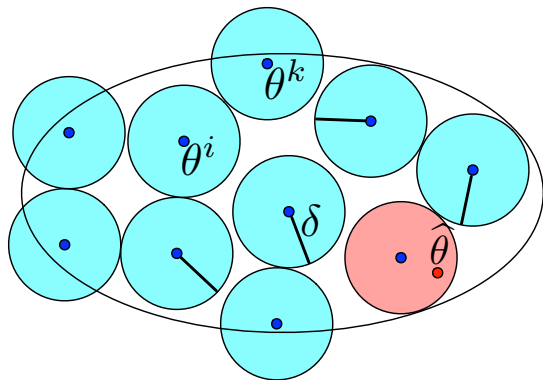
► Have 2δ -packing of Θ ,
i.e. $\{\theta^1, \theta^2, \dots, \theta^K\}$

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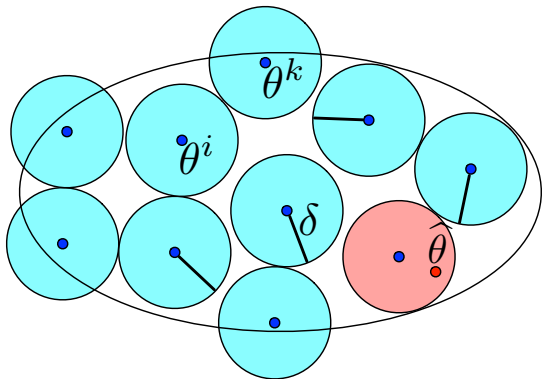
► Estimator $\hat{\theta}$

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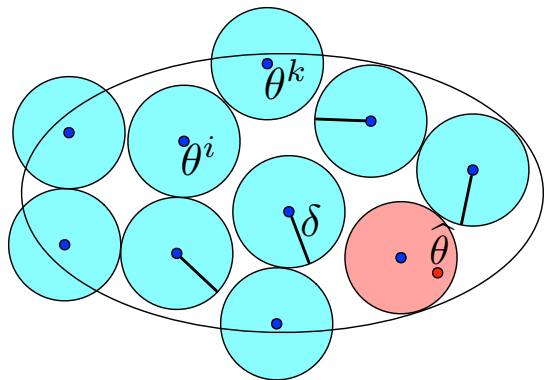
- ▶ Have 2δ -packing of Θ , i.e. $\{\theta^1, \theta^2, \dots, \theta^K\}$
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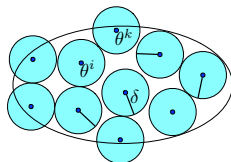
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- ▶ Can *test* index $i \in [K]$

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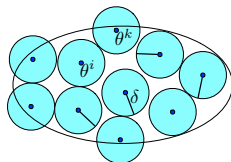
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- ▶ Nature chooses random index $V \in [K]$



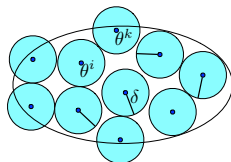
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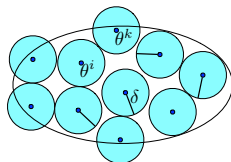


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- ▶ **Lower bound** minimax error:

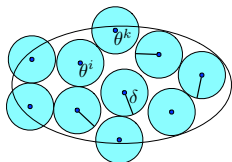
$$\begin{aligned} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[\rho(\hat{\theta}, \theta(P)) \right] &\geq \frac{1}{K} \sum_{v=1}^K \mathbb{E}_v \left[\rho(\hat{\theta}, \theta_v) \right] \\ &\geq \frac{1}{K} \sum_{v=1}^K \rho(\delta) P_v(\rho(\hat{\theta}, \theta_v) \geq 2\delta) \end{aligned}$$

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- ▶ Final **canonical testing problem**:

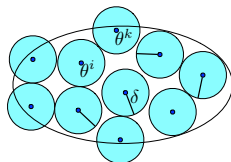
$$\mathfrak{M}_n(\theta(\mathcal{P}), \rho) \geq \rho(\delta) \min_{\hat{v}} \mathbb{P}(\hat{v}(X_1, \dots, X_n) \neq V).$$

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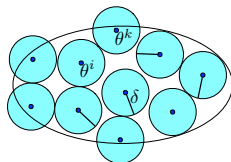


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Le Cam's method

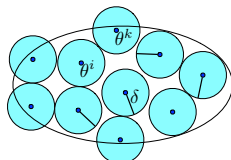
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Fano's inequality (for $K > 2$)

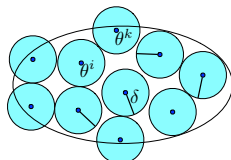
$$\begin{aligned} \mathbb{P}(\hat{v} \neq V) \\ \geq 1 - \frac{I(X_1, \dots, X_n; V) + \log 2}{\log K} \end{aligned}$$

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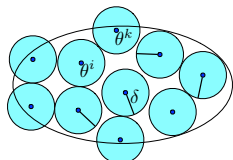
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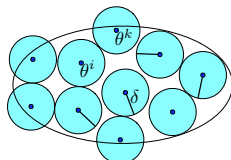


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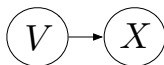
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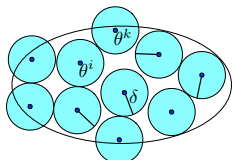


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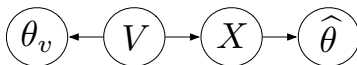
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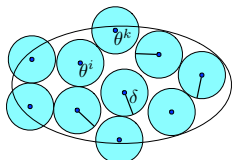


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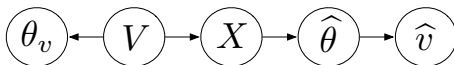
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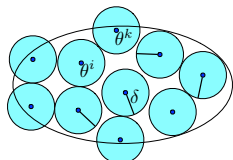


Proving minimax lower bounds

Step 1: Reduce from estimation to testing

Step 2: Canonical testing problem

Step 3: Classical bounds on error probabilities



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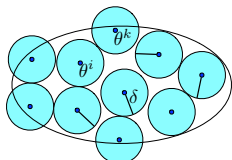


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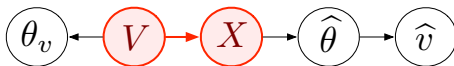
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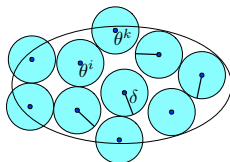
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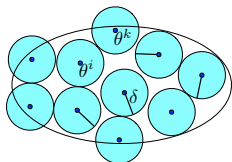
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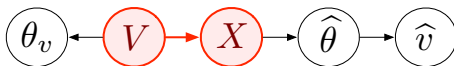
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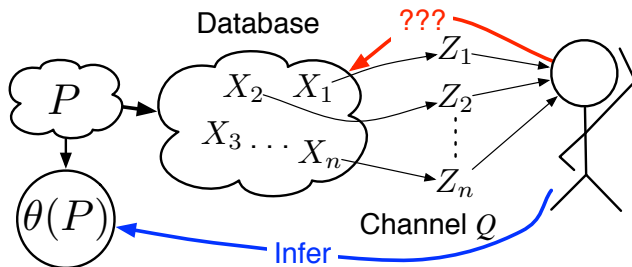
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Key idea: Control [information-theoretic divergences](#)

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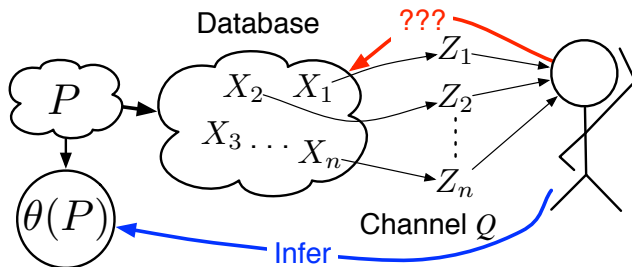
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Inference under privacy constraints



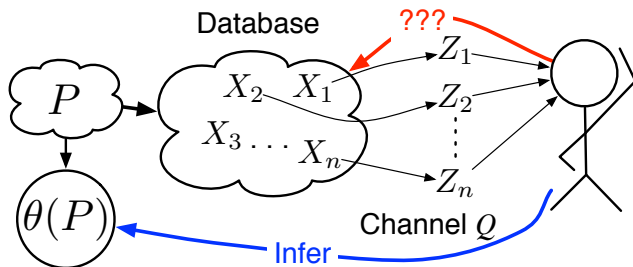
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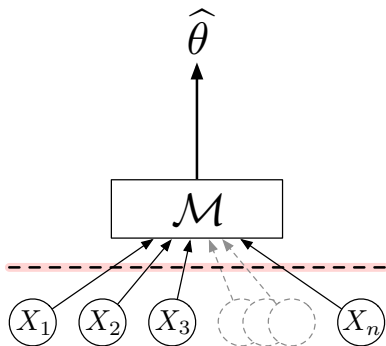
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Model of privacy

Local Privacy: Don't trust collector of data (Evfimievski et al. 2003, Warner 1965)

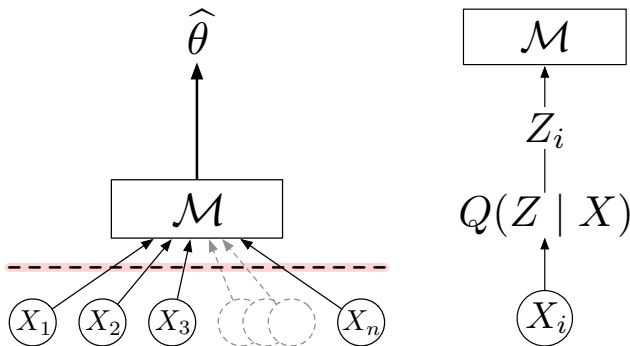
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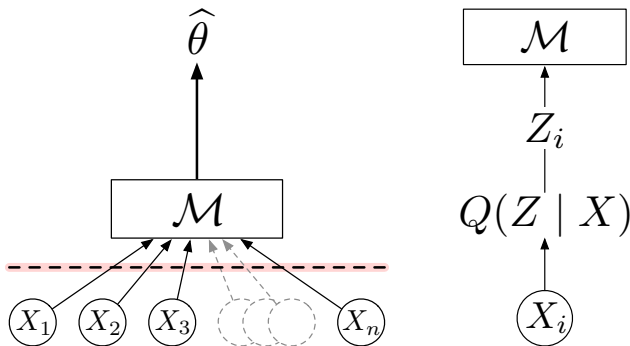
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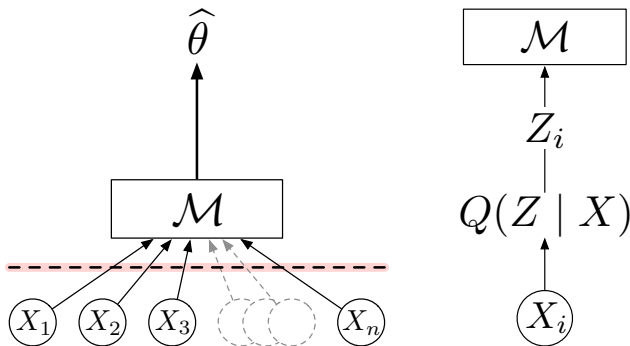
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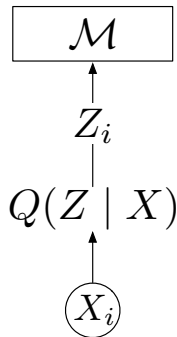
- ▶ Individuals $i \in \{1, \dots, n\}$ have personal data $X_i \sim P_\theta$
- ▶ Estimator $Z_1^n \mapsto \hat{\theta}(Z_{1:n})$

Differential privacy

Definition: The channel Q is α -differentially private if

$$\max_{z,x,x'} \frac{Q(Z = z \mid x)}{Q(Z = z \mid x')} \leq e^\alpha.$$

[Dwork, McSherry, Nissim, Smith 2006]

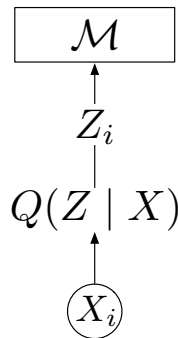
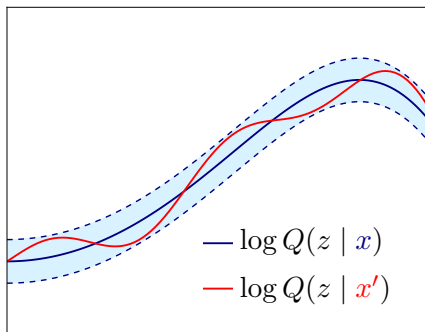


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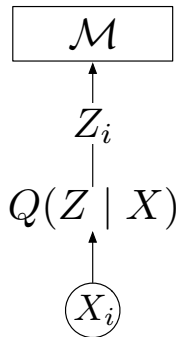
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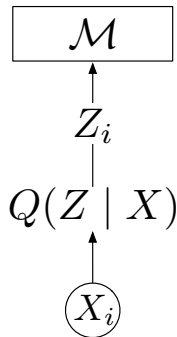
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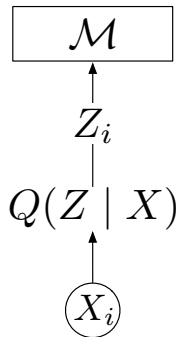
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$$\text{FNR} + \text{FPR} \geq \frac{2}{1 + e^\alpha}$$



[Wasserman and Zhou 2011]

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Central object of study: Minimax risk

- ▶ Parameter $\theta(P)$ of distribution P
- ▶ Loss ρ that measures error in estimate of $\hat{\theta}$ for θ : $\rho(\hat{\theta}, \theta)$
 - ▶ E.g. $\rho(\hat{\theta}, \theta) = \|\hat{\theta} - \theta\|_2^2$
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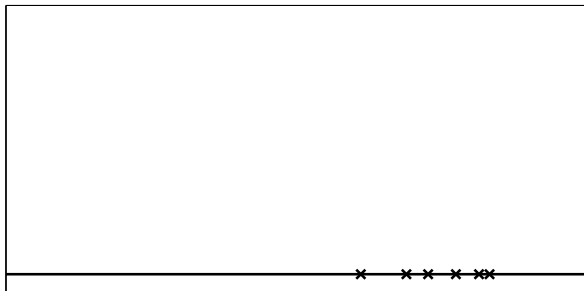
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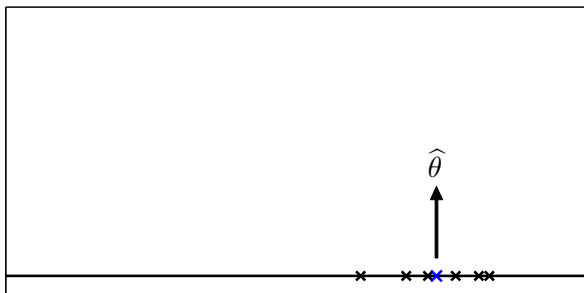
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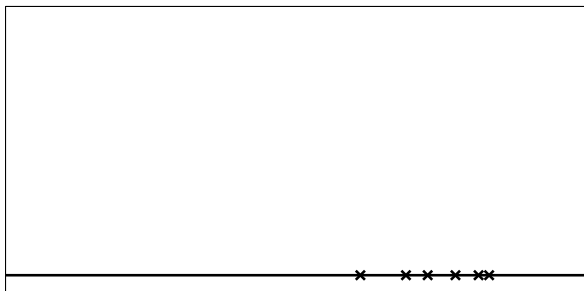
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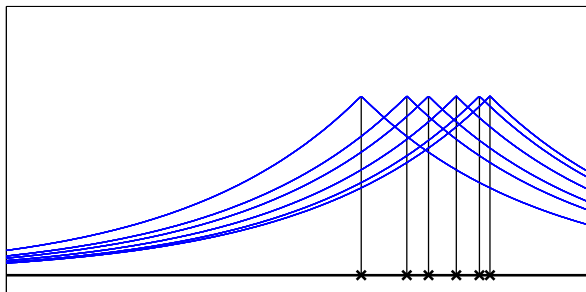
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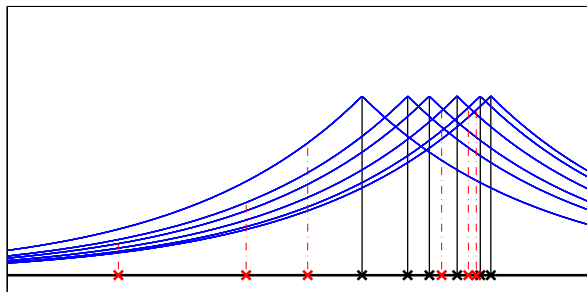
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▶ _____	\$0–\$10,000	$\theta_1 = .05$
▶ _____	\$10,001–\$20,000	$\theta_2 = .1$
▶ _____	\$20,001–\$40,000	$\theta_3 = .2$
▶ _____	\$40,001–\$80,000	$\theta_4 = .4$
▶ _____	\$80,001–\$160,000	$\theta_5 = .2$
▶ _____	\$160,001–\$320,000	$\theta_6 = .04$
▶ _____	\$320,001+	$\theta_7 = .01$

Example 2: multinomial estimation

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Usual rate:

$$\mathbb{E} \left[\|\hat{\theta} - \theta\|_2^2 \right] \leq \frac{1}{n}.$$

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Take away: Sample size reduction

$$n \mapsto \frac{n\alpha^2}{d}$$

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Goal: Understand tradeoff between differential privacy bound α and sample size n

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“Theorem 2” Effective sample size for d -dimensional problems scales as

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General theory

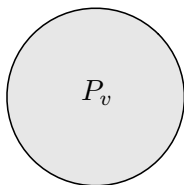
Showing minimax bounds:

- ▶ Have possible “true” parameters $\{\theta_v\}$ we want to find
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- ▶ Problem is *hard* when $P_v \approx P_{v'}$

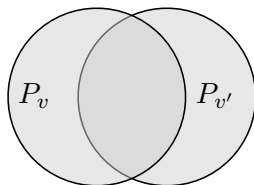
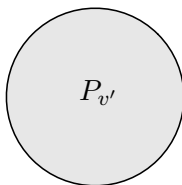
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Easy



Hard

Differential privacy and probability distributions

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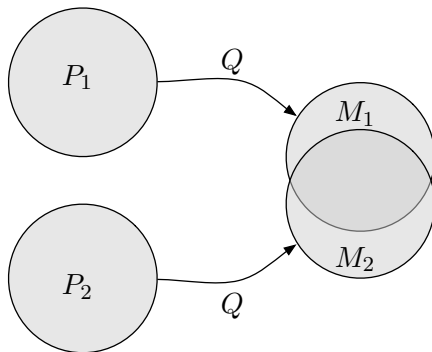
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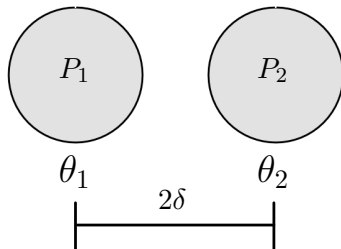
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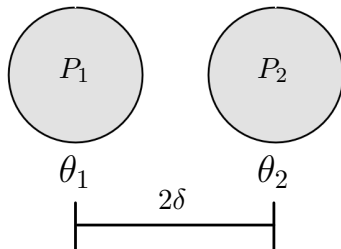
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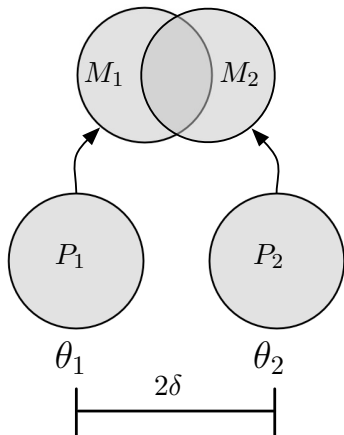
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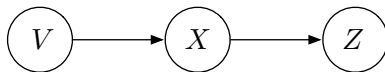
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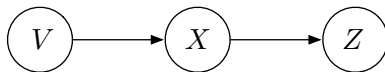
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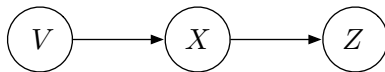


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Difficulty of problem: Saw earlier *mutual information*

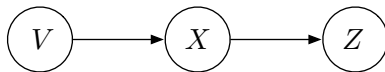
$$I(X_1, \dots, X_n; V) \mapsto I(Z_1, \dots, Z_n; V)$$

Fano inequality, lower bounds, contraction



- ▶ Have parameters $\theta_1, \dots, \theta_K$, choose randomly $V \in [K]$
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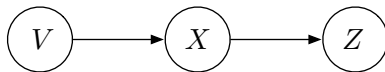
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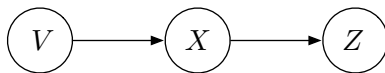
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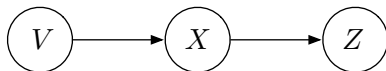
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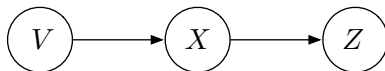
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Variational results on privacy and probability distributions



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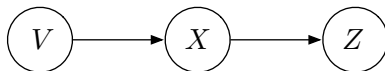


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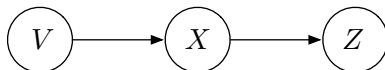


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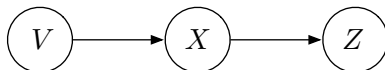
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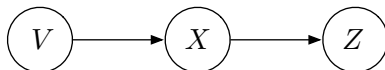
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Pre/post-print online: “Local privacy and statistical minimax rates.” D., Jordan, & Wainwright (2013). *arXiv:1302.3203 [stat.TH]*