Utility Invariance in Non-Cooperative Games

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Abstract:
Game theory traditionally specifies players' numerical payoff functions. Following the concept of utility invariance in modern social choice theory, this paper explores what is implied by specifying equivalence classes of utility function profiles instead. Within a single game, utility transformations that depend on other players' strategies preserve players' preferences over their own strategies, and so most standard non-cooperative solution concepts. Quantal responses and evolutionary dynamics are also considered briefly. Classical concepts of ordinal and cardinal non-comparable utility emerge when the solution is required to be invariant for an entire class of "consequentialist game forms" simultaneously.

JEL Classification: C70, D11, D71.

Keywords: Utility invariance, social welfare functional, ordinal utility, cardinal utility, two-person zero-sum games, weighted constant-sum games, weighted potential games, quantal responses, replicator dynamics, game forms.

Acknowledgements

A very preliminary version of some of the ideas discussed in this paper was presented to a seminar in May 2002 at the Università Cattolica del Sacro Cuore in Milan. My thanks to Luigi Campiglio for inviting me, and to the audience for their questions and comments. The present version (February 2005) is offered as a tribute to my friend and colleague Christian Seidl, with whom (and also Salvador Barberà) it has been my privilege to co-edit the *Handbook of Utility Theory*.

1 Introduction

Individual behaviour that maximizes utility is invariant under strictly increasing transformations of the utility function. In this sense, utility is ordinal. Similarly, behaviour under risk that maximizes expected utility is invariant under strictly increasing affine transformations of the utility function. In this sense, expected utility is cardinal. Following Sen (1976), a standard way to describe social choice with (or without) interpersonal comparisons is by means of a social welfare functional that maps each profile of individual utility functions to a social ordering. Different forms of interpersonal comparison, and different degrees of interpersonal comparability, are then represented by invariance under different classes of transformation applied to the whole profile of individual utility functions.[1]

This paper reports some results from applying a similar idea to non-cooperative games. That is, one asks what transformations of individual utility profiles have no effect on the relevant equilibrium set, or other appropriate solution concept.[2] So far, only a small literature has addressed this question, and largely in the context of cooperative or coalitional games.[3] For the sake of simplicity and brevity, this paper will focus on simple sufficient conditions for transformations to preserve some particular non-cooperative solution concepts.[4]

Before embarking on game theory, Section 2 begins with a brief review of relevant concepts in single person utility theory. Next, Section 3 describes the main invariance concepts that arise in social choice theory.

The discussion of games begins in Section 4 with a brief consideration of games with numerical utilities, when invariance is not an issue. Section 5 then considers concepts that apply only to pure strategies. Next, Section 6

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[1]See, for example, Sen (1974, 1977, 1979), d'Aspremont and Gevers (1977, 2002), Roberts (1980), Mongin and d'Aspremont (1998), Bossert and Weymark (2004).

[2]Of course, the usual definition of a game has each player's objective function described by a *payoff* rather than a utility function. Sometimes, this is the case of numerical utility described below. More often, it simply repeats the methodological error that was common in economics before Fisher (1892) and Pareto (1896) pointed out that an arbitrary increasing transformation of a consumer's utility function has no effect on demand.

[3]See especially Nash (1950), Shapley (1969), Roth (1979), Aumann (1985), Dagan and Serrano (1998), as well as particular parts of the surveys by Thomson (1994, pp. 1254–6), McLean (2002), and Kaneko and Wooders (2004). Recent contributions on "ordinal" bargaining theory include Kıbrıs (2004a, b) and Samet and Safra (2005).

[4]For a much more thorough discussion, especially of conditions that are both necessary and sufficient for transformations to preserve best or better responses, see Morris and Ui (2004).

moves on to mixed strategies. Thereafter, Sections 7 and 8 offer brief discussions of quantal responses and evolutionary dynamics.

Section 9 asks a different but related question: what kinds of transformation preserve equilibrium not just in a single game, but an entire class of game forms with outcomes in a particular consequence domain? In the end, this extended form of invariance seems much more natural than invariance for a single game. In particular, most of game theory can be divided into one part that considers only pure strategies, in which case it is natural to regard individuals' utilities as ordinal, and a second part that considers mixed strategies, in which case it is natural to regard individuals' utilities as cardinal. Neither case relies on any form of interpersonal comparison.

A few concluding remarks make up Section 10.

2 Single Person Decision Theory

2.1 Individual Choice and Utility

Let X be a fixed consequence domain. Let $\mathcal{F}(X)$ denote the family of non-empty subsets of X. Let R be any (complete and transitive) preference ordering on X. A *utility function representing* R is any mapping $u : X \to \mathbb{R}$ satisfying $u(x) \geq u(y)$ iff $x \; R \; y$, for every pair $x, y \in X$. Given any utility function $u : X \to \mathbb{R}$, for each *feasible set* $F \in \mathcal{F}(X)$, let

$$C(F, u) := \arg\max_x \{ \, u(x) \mid x \in F \, \} := \{ \, x^* \in F \mid x \in F \Longrightarrow u(x^*) \geq u(x) \, \}$$

denote the *choice set* of utility maximizing members of F. The mapping $F \mapsto\!\!\!\!\rightarrow C(F, u)$ is called the *choice correspondence* that is generated by maximizing u over each possible feasible set.[5]

A *transformation* of the utility function u is a mapping $\phi : \mathbb{R} \to \mathbb{R}$ that is used to generate an alternative utility function $\tilde{u} = \phi \circ u$ defined by $\tilde{u}(x) = (\phi \circ u)(x) := \phi[u(x)]$ for all $x \in X$.

2.2 Ordinal Utility

Two utility functions $u, \tilde{u} : X \to \mathbb{R}$ are said to be *ordinally equivalent* if and only if each of the following three equivalent conditions is satisfied:

[5]The decision theory literature usually refers to the mapping as a "choice function". The term "choice correspondence" accords better, however, with the terms "social choice correspondence" and "equilibrium correspondence" that are widespread in social choice theory and game theory, respectively. Also, it is often assumed that each choice set is non-empty, but this requirement will not be important in this paper.

(i) $u(x) \geq u(y)$ iff $\tilde{u}(x) \geq \tilde{u}(y)$, for all pairs $x, y \in X$;

(ii) $\tilde{u} = \phi \circ u$ for some strictly increasing transformation $\phi : \mathbb{R} \to \mathbb{R}$;

(iii) $C(F, u) = C(F, \tilde{u})$ for all $F \in \mathcal{F}(X)$.

Item (iii) expresses the fact that the *choice correspondence* defined by the mapping $F \mapsto\!\!\!\to C(F, u)$ on the domain $\mathcal{F}(X)$ must be invariant under strictly increasing transformations of the utility function u. Such transformations replace u by any member of the same ordinal equivalence class.

2.3 Lotteries and Cardinal Utility

A (simple) *lottery* on X is any mapping $\lambda : X \to \mathbb{R}_+$ such that:

(i) $\lambda(x) > 0$ iff $x \in S$, where S is the **finite** *support* of λ;

(ii) $\sum_{x \in X} \lambda(x) = \sum_{x \in S} \lambda(x) = 1$.

Thus $\lambda(x)$ is the probability that x is the outcome of the lottery. Let $\Delta(X)$ denote the set of all such simple lotteries.

Say that the preference ordering R on X is *von Neumann–Morgenstern* (or NM) if and only if there is a *von Neumann–Morgenstern* (or NM) *utility function* $v : X \to \mathbb{R}$ whose *expected value* $\mathbb{E}_\lambda v := \sum_{x \in X} \lambda(x) v(x) = \sum_{x \in S} \lambda(x) v(x)$ *represents* R on $\Delta(X)$. That is, $\mathbb{E}_\lambda v \geq \mathbb{E}_\mu v$ iff $\lambda \; R \; \mu$, for every pair $\lambda, \mu \in \Delta(X)$.

Let $\mathcal{F}_L(X) = \mathcal{F}(\Delta(X))$ denote the family of non-empty subsets of $\Delta(X)$. Given any $F \in \mathcal{F}_L(X)$ and any NM utility function $v : X \to \mathbb{R}$, let

$$C_L(F, u) := \arg\max_{\lambda} \{ \, \mathbb{E}_\lambda v \mid \lambda \in F \, \} := \{ \, \lambda^* \in F \mid \lambda \in F \Longrightarrow \mathbb{E}_{\lambda^*} v \geq \mathbb{E}_\lambda v \, \}$$

denote the *choice set* of expected utility maximizing members of F.

The mapping $\phi : \mathbb{R} \to \mathbb{R}$ is said to be a *strictly increasing affine transformation* if there exist an additive constant $\alpha \in \mathbb{R}$ and a positive multiplicative constant $\delta \in \mathbb{R}$ such that $\phi(r) \equiv \alpha + \delta r$. Two NM utility functions $v, \tilde{v} : X \to \mathbb{R}$ are said to be *cardinally equivalent* if and only if each of the following three equivalent conditions is satisfied:

(i) $\mathbb{E}_\lambda v \geq \mathbb{E}_\mu v$ iff $\mathbb{E}_\lambda \tilde{v} \geq \mathbb{E}_\mu \tilde{v}$, for all pairs $\lambda, \mu \in \Delta(X)$;

(ii) $\tilde{v} = \phi \circ v$ for some strictly increasing affine transformation $\phi : \mathbb{R} \to \mathbb{R}$;

(iii) $C_L(F, v) = C_L(F, \tilde{v})$ for all $F \in \mathcal{F}_L(X)$.

Item (iii) expresses the fact that the *lottery choice correspondence* defined by the mapping $F \mapsto\!\!\!\rightarrow C_L(F, v)$ on the lottery domain $\mathcal{F}_L(X)$ must be invariant under strictly increasing affine transformations of the utility function v. Such transformations replace v by any member of the same cardinal equivalence class.

3 Social Choice Correspondences

3.1 Arrow Social Welfare Functions

Let $\mathcal{R}(X)$ denote the set of preference orderings on X. A *preference profile* is a mapping $i \mapsto R_i$ from N to $\mathcal{R}(X)$ specifying the preference ordering of each individual $i \in N$. Let $R^N = \langle R_i \rangle_{i \in N}$ denote such a preference profile, and $\mathcal{R}^N(X)$ the set of all possible preference profiles.

Let $\mathcal{D} \subset \mathcal{R}^N(X)$ denote a *domain* of permissible preference profiles. A *social choice correspondence* (or SCC) can be represented as a mapping $(F, R^N) \mapsto\!\!\!\rightarrow C(F, R^N) \subset F$ from pairs in $\mathcal{F}(X) \times \mathcal{D}$ consisting of feasible sets and preference profiles to social choice sets. In the usual special case when $C(F, R^N)$ consists of those elements in F which maximize some social ordering R that depends on R^N, this SCC can be represented by an *Arrow social welfare function* (or ASWF) $f : \mathcal{D} \to \mathcal{R}(X)$ which maps each permissible profile $R^N \in \mathcal{D}$ to a social ordering $f(R^N)$ on X.

3.2 Social Welfare Functionals

Sen (1970) proposed extending the concept of an Arrow social welfare function by refining the domain to profiles of individual utility functions rather than preference orderings. This extension offers a way to represent different degrees of interpersonal comparability that might be embedded in the social ordering.

Formally, let $\mathcal{U}(X)$ denote the set of utility functions on X. A *utility function profile* is a mapping $i \mapsto u_i$ from N to $\mathcal{U}(X)$. Let $u^N = \langle u_i \rangle_{i \in N}$ denote such a profile, and $\mathcal{U}^N(X)$ the set of all possible utility function profiles.

Given a domain $\mathcal{D} \subset \mathcal{U}^N(X)$ of permissible utility function profiles, an SCC is a mapping $(F, u^N) \mapsto\!\!\!\rightarrow C(F, u^N) \subset F$ defined on $\mathcal{F}(X) \times \mathcal{D}$. When each choice set $C(F, u^N)$ consists of elements $x \in F$ that maximize a social ordering R, there is a *social welfare functional* (or SWFL) $G : \mathcal{D} \to \mathcal{R}(X)$ mapping $\mathcal{D} \subset \mathcal{U}^N(X)$ to the set of possible social orderings.

3.3 Utility Invariance in Social Choice

Given a specific SCC $(F, u^N) \mapsto\!\!\!\rightarrow C(F, u^N)$, one can define an equivalence relation $\sim$ on the space of utility function profiles $\mathcal{U}^N(X)$ by specifying that $u^N \sim \tilde{u}^N$ if and only if $C(F, u^N) = C(F, \tilde{u}^N)$ for all $F \in \mathcal{F}(X)$.

An *invariance transformation* of the utility function profiles is a profile $\phi^N = \langle \phi_i \rangle_{i \in N}$ of individual utility transformations $\phi_i : \mathbb{R} \to \mathbb{R}$ having the property that $u^N \sim \tilde{u}^N$ whenever $\tilde{u}^N = \phi^N(u^N)$ — i.e., $\tilde{u}_i = \phi_i(u_i)$ for all $i \in N$. Thus, invariance transformations result in equivalent utility function profiles, for which the SCC generates the same choice set. In the following, let Φ denote the class of invariance transformations.

3.3.1 Ordinal Non-Comparability

The first specific concept of utility invariance for SCCs arises when Φ consists of mappings $\phi^N = \langle \phi_i \rangle_{i \in N}$ from $\mathbb{R}^N$ into itself with the property that each $\phi_i : \mathbb{R} \to \mathbb{R}$ is strictly increasing. So $u^N \sim \tilde{u}^N$ if and only if the two profiles $u^N, \tilde{u}^N$ have the property that the utility functions $u_i, \tilde{u}_i$ of each individual $i \in N$ are ordinally equivalent. The SCC $C(F, u^N)$ is said to satisfy *ordinal non-comparability* (or ONC) if $C(F, u^N) = C(F, \tilde{u}^N)$ for all $F \in \mathcal{F}(X)$ whenever u^N and $\tilde{u}^N$ are ordinally equivalent in this way.

Obviously, in this case each equivalence class of an individual's utility functions is represented by one corresponding preference ordering, and each equivalence class of utility function profiles is represented by one corresponding preference profile. So the SCC can be expressed in the form $C^*(F, R^N)$. In particular, if each social choice set $C(F, u^N)$ maximizes a social ordering, implying that there is a SWFL, then that SWFL takes the form of an Arrow social welfare function.

3.3.2 Cardinal Non-Comparability

The second specific concept of utility invariance arises when Φ consists of mappings $\phi^N = \langle \phi_i \rangle_{i \in N}$ from $\mathbb{R}^N$ into itself with the property that each $\phi_i : \mathbb{R} \to \mathbb{R}$ is strictly increasing and affine. That is, there must exist additive constants α_i and positive multiplicative constants δ_i such that $\phi_i(r) = \alpha_i + \delta_i r$ for each $i \in N$ and all $r \in \mathbb{R}$. The SCC $C(F, u^N)$ is said to satisfy *cardinal non-comparability* (or CNC) when it meets this invariance requirement.

3.3.3 Ordinal Level Comparability

Interpersonal comparisons of utility levels take the form $u_i(x) > u_j(y)$ or $u_i(x) < u_j(y)$ or $u_i(x) = u_j(y)$ for a pair of individuals $i, j \in N$ and a pair of social consequences $x, y \in N$. Such comparisons will not be preserved when different increasing transformations ϕ_i and ϕ_j are applied to i's and j's utilities. Indeed, level comparisons are preserved, in general, only if the same transformation is applied to all individuals' utilities.

Accordingly, in this case the invariance class Φ consists of those mappings $\phi^N = \langle \phi_i \rangle_{i \in N}$ for which there exists a strictly increasing transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi_i = \phi$ for all $i \in N$. An SCC with this invariance class is said to satisfy *ordinal level comparability* (or OLC).

3.3.4 Cardinal Unit Comparability

Comparisons of utility sums take the form $\sum_{i \in N} u_i(x) > \sum_{i \in N} u_i(y)$ or $\sum_{i \in N} u_i(x) < \sum_{i \in N} u_i(y)$ or $\sum_{i \in N} u_i(x) = \sum_{i \in N} u_i(y)$ for a pair of social consequences $x, y \in N$. Such comparisons rely on being able to compare different individuals' utility differences, so one can say that one person's gain outweighs another person's loss. Such comparisons are only preserved, in general, if and only if increasing *co-affine* transformations are applied to all individuals' utilities. That is, the mappings ϕ_i $(i \in N)$ must take the form $\phi_i(r) = \alpha_i + \delta r$ for suitable additive constants α_i $(i \in N)$, and a positive multiplicative constant δ that is independent of i. An SCC with this invariance class is said to satisfy *cardinal unit comparability* (or CUC).

3.3.5 Cardinal Full Comparability

Some welfare economists, following a suggestion of Sen (1973), have looked for income distributions that equalize utility levels while also maximizing a utility sum. In order that comparisons of both utility sums and utility levels should be invariant, the mappings ϕ_i $(i \in N)$ must take the form $\phi_i(r) = \alpha + \delta r$ for a suitable additive constant α and a positive multiplicative constant δ that are both independent of i. An SCC with this invariance class is said to satisfy *cardinal full comparability* (or CFC).

3.3.6 Cardinal Ratio Scales

In discussions of optimal population, the utility sum $\sum_{i \in N} u_i(x)$ may get replaced by $\sum_{i \in M} u_i(x)$ for a subset $M \subset N$ of "relevant" individuals. Presumably, the individuals $i \in N \setminus M$ never come into existence, and so have their utilities set to zero by a convenient normalization. This form of welfare sum, with the set of individuals M itself subject to choice, allows comparisons of extended social states (M, x) and (M', x') depending on which of the two sums $\sum_{i \in M} u_i(x)$ and $\sum_{i \in M'} u_i(x')$ is greater. These comparisons are preserved when the mappings ϕ_i $(i \in N)$ take the form $\phi_i(r) = \rho r$ for a positive multiplicative constant ρ that is independent of i. An SCC with this invariance class is said to satisfy *cardinal ratio scale comparability* (or CRSC).

4 Games with Numerical Utility

4.1 Games in Normal Form

A *game in normal form* is a triple $G = \langle N, S^N, u^N \rangle$ where:

(i) N is a finite set of *players*;

(ii) each player $i \in N$ has a *strategy set* S_i, and $S^N = \prod_{i \in N} S_i$ is the set of *strategy profiles*;

(iii) each player $i \in N$ has a *utility function* $u_i : S^N \to \mathbb{R}$ defined on the domain of strategy profiles, and $u^N = \langle u_i \rangle_{i \in N}$ is the *utility profile*.

Of course game theorists usually refer to "payoff functions" instead of utility functions. But the whole point of this paper is to see what properties such functions share with the utility functions that ordinarily arise in single person decision theory. To emphasize this comparison, the term "utility functions" will be used in games just as it is in decision theory and in social choice theory.

4.2 Numerical Utility

Before moving on to various forms of ordinal and cardinal utility in games, let us first readily concede that sometimes players in a game do have objectives that can be described simply by real numbers. For example, the players may be firms seeking to maximize profit, measured in a particular currency unit such as dollars, euros, yen, pounds, crowns, ... Or, as suggested by the title of von Neumann's (1928) classic paper, they may be people playing *Gesellschaftsspiele* (or "parlour games") for small stakes. The fact that players' objectives are then straightforward to describe is one reason why it is easier to teach undergraduate students producer theory before facing them with the additional conceptual challenges posed by consumer theory.

Even in these special settings, however, numerical utility should be seen as a convenient simplification that abstracts from many important aspects of reality. For example, most decisions by firms generate profits at different times and in different uncertain events. Standard decision theory requires that these profits be aggregated into a single objective. Finding the expected present discounted value might seem one way to do this, but it may not be obvious what are the appropriate discount factors or the probabilities of different events. As for any parlour game, the fact that people choose to play at all reveals either an optimistic assessment of their chances of winning, or probably more realistically, an enjoyment of the game that is not simply represented by monetary winnings, even if the game is played for money.

4.3 Some Special Games

4.3.1 Zero-Sum and Constant Sum Games

Much of the analysis in von Neumann (1928) and in von Neumann and Morgenstern (1944) is devoted to *two-person zero sum* games, in which the set of players is $N = \{1, 2\}$, and $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ for all $(s_1, s_2) \in S_1 \times S_2$. Von Neumann and Morgenstern (1944) also consider *n-person zero sum games* in which N remains a general finite set, and $\sum_{i \in N} u_i(s^N) \equiv 0$. They also argue that such games are equivalent to *constant sum games* in which $\sum_{i \in N} u_i(s^N) \equiv C$ for a suitable constant $C \in \mathbb{R}$.

4.3.2 Team Games

Following Marschak and Radner (1972), the game $G = \langle N, S^N, u^N \rangle$ is said to be a *team game* if there exists a single utility function $u^* : S^N \to \mathbb{R}$ with the property that $u_i \equiv u^*$ for all $i \in N$.

4.4 Beyond Numerical Utility

All of the definitions in this section are clear and familiar when players have numerical utilities. One of our tasks in later sections will be to investigate extensions of these concepts which apply to different kinds of ordinal or cardinal utility.

5 Games with Ordinal Utility

5.1 Ordinal Non-Comparability

A game $G = \langle N, S^N, u^N \rangle$ will be described as having *ordinal utility* if it is equivalent to each alternative game $\tilde{G} = \langle N, S^N, \tilde{u}^N \rangle$ with the same sets of players and strategies, but with transformed utility functions $\tilde{u}_i$ having the property that u_i and $\tilde{u}_i$ are ordinally equivalent for each $i \in N$. Thus, the players' utility functions are ordinally non-comparable. This section shows that most familiar concepts concerning pure strategies in non-cooperative games not only satisfy ordinal non-comparability because they are invariant under increasing transformations of individuals' utility functions, but are actually invariant under a much broader class of utility transformations.

5.1.1 Two-Person Strictly Competitive Games

A two-person game with $N = \{1, 2\}$ is said to be *strictly competitive* provided that for all $(s_1, s_2), (s'_1, s'_2) \in S_1 \times S_2$, one has $u_1(s_1, s_2) \geq u_1(s'_1, s'_2)$ iff $u_2(s_1, s_2) \leq u_2(s'_1, s'_2)$. Alternatively, the two players are said to have *opposing interests*.

Note that a two-person game is strictly competitive if and only if the utility function u_2 is ordinally equivalent to $-u_1$ — which is true iff u_1 is ordinally equivalent to $-u_2$. Thus, strict competitiveness is necessary and sufficient for a two-person game to be ordinally equivalent to a zero-sum game.

5.1.2 Ordinal Team Games

The game $G = \langle N, S^N, u^N \rangle$ is said to be an *ordinal team game* if there exists a single ordering R^* on S^N with the property that, for all $s^N, \tilde{s}^N \in S^N$, one has $s^N \; R^* \; \tilde{s}^N$ iff $u_i(s^N) \geq u_i(\tilde{s}^N)$ for all $i \in N$. Thus, all the players have ordinally equivalent utility functions. Often game theorists have preferred alternative terms such as "pure coordination game", or games with "common" or "identical interests".

The players all agree how to order different strategy profiles. So all agree what is the set of optimal strategy profiles, which are the only Pareto efficient profiles. Any Pareto efficient profile is a Nash equilibrium, but there can be multiple Nash equilibria. These multiple equilibria may be "Pareto ranked", in the sense that some equilibria are Pareto superior to others.

5.2 Pure Strategy Dominance and Best Replies

5.2.1 Strategy Contingent Preferences

Suppose player i faces known strategies s_j $(j \in N \setminus \{i\})$ chosen by the other players. Let $s_{-i} = \langle s_j \rangle_{j \in N \setminus \{i\}}$ denote the profile of these other players' strategies, and let $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$ be the set of all such profiles.

Given the utility function u_i on S^N, player i has a (strategy contingent) preference ordering $R_i(s_{-i})$ on S_i defined by

$$s_i \; R_i(s_{-i}) \; s_i' \Longleftrightarrow u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Let $P_i(s_{-i})$ denote the corresponding strict preference relation, which satisfies $s_i \; P_i(s_{-i}) \; s_i'$ iff $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

5.2.2 Domination by Pure Strategies

Player i's strategy $s_i \in S_i$ is *strictly dominated* by $s_i' \in S_i$ iff $s_i' \; P_i(\bar{s}_{-i}) \; s_i$ for all strategy profiles $\bar{s}_{-i} \in S_{-i}$ of the other players. Similarly, player i's strategy $s_i \in S_i$ is *weakly dominated* by $s_i' \in S_i$ iff $s_i' \; R_i(\bar{s}_{-i}) \; s_i$ for all strategy profiles $\bar{s}_{-i} \in S_{-i}$ of the other players, with $s_i \; P_i(s_{-i}) \; s_i'$ for at least one $s_{-i} \in S_{-i}$.

5.2.3 Best Replies

Given the utility function u_i on S^N, player i's set of *best replies* to s_{-i} is given by

$$\begin{aligned} B_i(s_{-i}; u_i) &:= \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}) \\ &= \{ \, s_i^* \in S_i \mid s_i \in S_i \Longrightarrow s_i^* \; R_i(s_{-i}) \; s_i \, \} \end{aligned}$$

The mapping $s_{-i} \mapsto\!\!\!\to B_i(s_{-i}; u_i)$ from S_{-i} to S_i is called player i's *best reply correspondence* $B_i(\cdot; u_i)$ given the utility function u_i.

5.2.4 Pure Strategy Solution Concepts

Evidently, each player i's family of preference orderings $R_i(s_{-i})$ $(s_{-i} \in S_{-i})$ determines which pure strategies dominate other pure strategies either weakly or strictly, as well as the best responses. It follows that the same orderings determine any solution concept which depends on dominance relations or best responses, such as Nash equilibrium, and strategies that survive iterated deletion of strategies which are dominated by other pure strategies.

5.3 Beyond Ordinal Non-Comparability

Each player i's family of preference orderings $R_i(s_{-i})$ $(s_{-i} \in S_{-i})$ over S_i is obviously invariant under utility transformations of the form $\tilde{u}_i(s^N) \equiv \psi_i(u_i(s^N); s_{-i})$ where for each fixed $\bar{s}_{-i} \in S_{-i}$ the mapping $r \mapsto \psi_i(r; \bar{s}_{-i})$ from $\mathbb{R}$ into itself is strictly increasing. This form of utility invariance is like the social choice property of ONC invariance for a society in which the set of individuals expands from N to the new set

$$N^* := \{ (i, s_{-i}) \mid i \in N,\ s_{-i} \in S_{-i} \} = \bigcup_{i \in N} [\{i\} \times S_{-i}]$$

The elements of this expanded set are "strategy contingent" versions of each player $i \in N$, whose existence depends on which strategy profile s_{-i} the other players choose. The best reply and equilibrium correspondences will accordingly be described as satisfying *strategy contingent ordinal non-comparability* (or SCONC).

5.4 Some Special Ordinal Games

5.4.1 Generalized Strictly Competitive Games

The two-person game $G = \langle N, S^N, u^N \rangle$ with $N = \{1, 2\}$ is said to be a *generalized strictly competitive game* if there exist strictly increasing strategy contingent transformations $r \mapsto \psi_1(r; \bar{s}_2)$ and $r \mapsto \psi_2(r; \bar{s}_1)$ of the two players' utility functions such that the transformed game $\tilde{G} = \langle N, S^N, \tilde{u}^N \rangle$ with $\tilde{u}_1(s_1, s_2) \equiv \phi_i(u_1(s_1, s_2), s_2)$ and $\tilde{u}_2(s_1, s_2) \equiv \phi_i(u_2(s_1, s_2), s_1)$ is a two-person zero-sum game. A necessary and sufficient condition for this property to hold is that the binary relation $\hat{R}$ on $S_1 \times S_2$ defined by

$$(s_1, s_2)\ \hat{R}\ (s_1', s_2') \iff \begin{array}{rcl} s_2 = s_2' & \text{and} & u_1(s_1, s_2) \geq u_1(s_1', s_2') \\ \text{or } s_1 = s_1' & \text{and} & u_2(s_1, s_2) \leq u_2(s_1', s_2') \end{array}$$

should admit a transitive extension.

5.4.2 Generalized Ordinal Team Games

The game $G = \langle N, S^N, v^N \rangle$ is said to be a *generalized ordinal team game* if there exists a single ordering R^* on S^N with the property that, for all $i \in N$, all $s_i, s_i' \in S_i$ and all $\bar{s}_{-i} \in S_{-i}$, one has $s_i\ R_i(\bar{s}_{-i})\ s_i'$ iff $(s_i, \bar{s}_{-i})\ R^*\ (s_i', \bar{s}_{-i})$. In any such game, the best reply correspondences and Nash equilibrium set will be identical to those in an ordinal team game.

6 Games with Cardinal Utility

6.1 Cardinal Non-Comparability

A game $G = \langle N, S^N, v^N \rangle$ will be described as having *cardinal utility* if it is equivalent to each alternative game $\tilde{G} = \langle N, S^N, \tilde{v}^N \rangle$ with the same sets of players and strategies, but with transformed utility functions $\tilde{v}_i$ having the property that v_i and $\tilde{v}_i$ are cardinally equivalent for each $i \in N$. That is, there must exist additive constants α_i and positive multiplicative constants δ_i such that $\tilde{v}_i(s^N) \equiv \alpha_i + \delta_i v_i(s^N)$ for all $i \in N$.

6.1.1 Zero and Constant Sum Games

A game $G = \langle N, S^N, v^N \rangle$ is said to be *zero sum* provided that $\sum_{i \in N} v_i(s^N) \equiv 0$. This property is preserved under increasing affine transformations $v_i \mapsto \alpha_i + \delta v_i$ where the multiplicative constant δ is independent of i, and the additive constants α_i satisfy $\sum_{i \in N} \alpha_i = 0$. Thus, the zero sum property relies on a strengthened form of the CUC invariance property described in Section 3.3.4.

The *constant sum property* is satisfied when there exists a constant $C \in \mathbb{R}$ such that $\sum_{i \in N} v_i(s^N) \equiv C$. This property is preserved under increasing affine transformations $v_i \mapsto \alpha_i + \delta v_i$ where the multiplicative constant δ is independent of i, and the additive constants α_i are arbitrary. Thus, the constant sum property is preserved under precisely the class of transformations allowed by the CUC invariance property. In this sense, the constant sum property relies on interpersonal comparisons of utility.

6.1.2 Constant Weighted Sum Games

Rather more interesting is the *constant weighted sum property*, which holds when there exist multiplicative weights ω_i ($i \in N$) and a constant $C \in \mathbb{R}$ such that $\sum_{i \in N} \omega_i v_i(s^N) \equiv C$. This property is preserved under all increasing affine transformations $v_i \mapsto \tilde{v}_i = \alpha_i + \delta_i v_i$ because, if $\sum_{i \in N} \omega_i v_i(s^N) \equiv C$, then $\sum_{i \in N} \tilde{\omega}_i \tilde{v}_i(s^N) \equiv \tilde{C}$ where $\tilde{\omega}_i = \omega_i / \delta_i$ and $\tilde{C} = C + \sum_{i \in N} \omega_i \alpha_i / \delta_i$. Thus, we are back in the case of CNC invariance, without interpersonal comparisons.

A two-person game with cardinal utilities is said to be *strictly competitive* if and only if the utility function u_2 is cardinally equivalent to $-u_1$. That is, there must exist a constant α and a positive constant δ such that $u_2(s_1, s_2) \equiv \alpha - \delta u_1(s_1, s_2)$. This form of strict competitiveness is therefore satisfied if and only if the two-person game with cardinal utilities has a constant weighted sum. The same condition is also necessary and sufficient for a two-person game to be cardinally equivalent to a zero-sum game.

6.1.3 Cardinal Team Games

The game $G = \langle N, S^N, v^N \rangle$ is said to be an *cardinal team game* if there exists a single utility function v^* on S^N which is cardinally equivalent to each player's utility function v_i. Thus, all the players must have cardinally equivalent utility functions, and so identical preferences over the space of lotteries $\Delta(S^N)$.

6.2 Dominated Strategies and Best Responses

6.2.1 Belief Contingent Preferences

Suppose player i attaches a probability $\pi_i(s_{-i})$ to each profile $s_{-i} \in S_{-i}$ of other players' strategies. That is, player i has *probabilistic beliefs* specified by π_i in the set $\Delta(S_{-i})$ of all probability distributions on the (finite) set S_{-i}.

Given any NM utility function v_i for player i defined on S^N, and given beliefs $\pi_i \in \Delta(S_{-i})$, let

$$V_i(s_i; \pi_i) := \sum_{s_{-i} \in S_{-i}} \pi_i(s_{-i}) v_i(s_i, s_{-i})$$

denote the expected value of v_i when player i chooses the pure strategy s_i. Then

$$\mathbb{E}_{\sigma_i} V_i(\cdot; \pi_i) := \sum_{s_i \in S_i} \sigma_i(s_i) V_i(s_i; \pi_i)$$

is the expected value of v_i when player i chooses the mixed strategy $\sigma_i \in \Delta(S_i)$. There is a corresponding (belief contingent) preference ordering $R_i(\pi_i)$ on $\Delta(S_i)$ for player i defined by

$$\sigma_i \; R_i(\pi_i) \; \sigma_i' \Longleftrightarrow \mathbb{E}_{\sigma_i} V_i(\cdot; \pi_i) \geq \mathbb{E}_{\sigma_i'} V_i(\cdot; \pi_i).$$

6.2.2 Dominated Strategies

Player i's strategy $s_i \in S_i$ is *strictly dominated* iff there exists an alternative mixed strategy $\sigma_i \in \Delta(S_i)$ such that $\sum_{\tilde{s}_i \in S_i} \sigma_i(\tilde{s}_i) v_i(\tilde{s}_i, \bar{s}_{-i}) > v_i(s_i, \bar{s}_{-i})$ for all strategy profiles $\bar{s}_{-i} \in S_{-i}$ of the other players. As is well known, a strategy may be strictly dominated even if there is no alternative pure strategy that dominates it. So the definition is less stringent than the one used for pure strategies.

Similarly, player i's strategy $s_i \in S_i$ is *weakly dominated* iff there exists an alternative mixed strategy $\sigma_i \in \Delta(S_i)$ such that $\sum_{\tilde{s}_i \in S_i} \sigma_i(\tilde{s}_i) v_i(\tilde{s}_i, \bar{s}_{-i}) \geq v_i(s_i, \bar{s}_{-i})$ for all strategy profiles $\bar{s}_{-i} \in S_{-i}$ of the other players, with strict inequality for at least one such strategy profile.

6.2.3 Best Replies

Given the NM utility function v_i, player i's set of *best replies* to π_i is

$$B_i(\pi_i; v_i) := \arg\max_{s_i \in S_i} V_i(s_i; \pi_i)$$

The mapping $\pi_i \mapsto\!\!\!\rightarrow B_i(\pi_i; v_i)$ from $\Delta(S_{-i})$ to S_i is called player i's *best reply correspondence* $B_i(\cdot; v_i)$ given the NM utility function v_i. It is easy to see that the set

$$\{\, \sigma_i^* \in \Delta(S_i) \mid \sigma_i \in \Delta(S_i) \Longrightarrow \sigma_i^* \; R_i(\pi_i) \; \sigma_i \,\}$$

of *mixed strategy best replies* to π_i is equal to $\Delta(B_i(\pi_i; v_i))$, the subset of those $\sigma_i \in \Delta(S_i)$ that satisfy $\sum_{s_i \in B_i(\pi_i; v_i)} \sigma_i(s_i) = 1$.

6.3 Beyond Cardinal Non-Comparability

The definitions above evidently imply that the preferences $R_i(\pi_i)$ and the set of player i's dominated strategies are invariant under increasing affine transformations of the form

$$\tilde{v}_i(s^N) \equiv \alpha_i(s_{-i}) + \delta_i v_i(s^N)$$

where, for each $i \in N$, the multiplicative constant δ_i is positive. So, of course, are each player's best reply correspondence $B_i(\cdot; v_i)$, as well as the sets of Nash equilibria, correlated equilibria, and rationalizable strategies.[6] This property will be called *strategy contingent cardinal non-comparability* — or SCCNC invariance.

As in the case of SCONC invariance discussed in Section 5.3, consider the expanded set

$$N^* := \{\, (i, s_{-i}) \mid i \in N,\ s_{-i} \in S_{-i} \,\} = \bigcup_{i \in N} [\{i\} \times S_{-i}]$$

[6] Such non-cooperative solution concepts are defined and discussed in Hammond (2004), as well as some in the game theory textbooks cited there — for example, Fudenberg and Tirole (1991) or Osborne and Rubinstein (1994).

of strategy contingent versions of each player $i \in N$. Then SCCNC invariance amounts to CUC invariance between members of the set $N_i^* := \{ (i, s_{-i}) \mid s_{-i} \in S_{-i} \}$, for each $i \in N$, combined with CNC invariance between members of different sets N_i^*.

6.4 Some Special Cardinal Games

6.4.1 Generalized Zero-Sum Games

The two-person game $G = \langle N, S^N, v^N \rangle$ with $N = \{1, 2\}$ is said to be a *two-person generalized zero-sum game* if there exist strictly increasing strategy contingent affine transformations of the form described in Section 6.3 — namely,

$$\begin{aligned} \tilde{v}_1(s_1, s_2) &\equiv \alpha_1(s_2) + \rho_1 v_1(s_1, s_2) \\ \text{and } \tilde{v}_2(s_1, s_2) &\equiv \alpha_2(s_1) + \rho_2 v_2(s_1, s_2) \end{aligned}$$

— such that $\tilde{v}_1 + \tilde{v}_2 \equiv 0$. This will be true if and only if

$$v_2(s_1, s_2) \equiv -\rho v_1(s_1, s_2) + \alpha_2^*(s_1) + \alpha_1^*(s_2)$$

for suitable functions $\alpha_2^*(s_1)$, $\alpha_1^*(s_2)$, and a suitable positive constant ρ. These transformations are more general than those allowed in the constant weighted sum games of Section 6.1.2 because the additive constants can depend on the other player's strategy.

For ordinary two-person zero-sum games there are well known special results such as the maximin theorem, and special techniques such as linear programming. Obviously, one can adapt these to two-person generalized zero-sum games.

6.4.2 Generalized Cardinal Team Games

The game $G = \langle N, S^N, v^N \rangle$ is said to be a *generalized cardinal team game* if each player's utility function can be expressed as an increasing strategy contingent affine transformation $v_i(s^N) \equiv \alpha_i(s_{-i}) + \rho_i v^*(s^N)$ of a common cardinal utility function v^*. Then the best reply correspondences and Nash equilibrium set will be identical to those in the cardinal team game with this common utility function.

In such a game, note that i's gain to deviating from the strategy profile s^N by choosing s_i' instead is given by

$$v_i(s_i', s_{-i}) - v_i(s_i, s_{-i}) = \rho_i [v^*(s_i', s_{-i}) - v^*(s_i, s_{-i})].$$

In the special case when $\rho_i = 1$ for all $i \in N$, this implies that G is a *potential game*, with v^* as the *potential function*. Because of this restriction on the constants ρ_i, however, this definition due to Monderer and Shapley (1996) involves implicit interpersonal comparisons. See Ui (2000) in particular for further discussion of potential games. Morris and Ui (2005) describe games with more general constants ρ_i as *weighted potential games*. They also consider *generalized potential games* which are best response equivalent to cardinal team games.

7 Quantal Response Equilibria

7.1 Stochastic Utility

Given any feasible set $F \in \mathcal{F}$, ordinary decision theory considers a choice set $C(F) \subset F$. On the other hand, *stochastic decision theory* considers a simple *choice lottery* $q(\cdot, F) \in \Delta(F)$ defined for each $F \in \mathcal{F}$. Specifically, let $q(x, F)$ denote the probability of choosing $x \in F$ when the agent is presented with the feasible set F.

Following the important choice model due to Luce (1958, 1959), the mapping $u : X \to \mathbb{R}_+$ is said to be a *stochastic utility function* in the case when $q(x, F) = u(x)/\sum_{y \in F} u(y)$ for all $x \in F$. In this case $q(x, F)$ is obviously invariant to transformations of u that take the form $\tilde{u}(x) \equiv \rho u(x)$ for a suitable mutiplicative constant $\rho > 0$. Thus, u is a positive-valued function defined up to a ratio scale. And whenever $x, y \in F \in \mathcal{F}$, the utility ratio $u(x)/u(y)$ becomes equal to the choice probability ratio $q(x, F)/q(y, F)$.

Much econometric work on discrete choice uses the special *multinomial logit* version of Luce's model, in which $\ln u(x) \equiv \beta U(x)$ for a suitable *logit utility* function U on $\Delta(Y)$ and a suitable constant $\beta > 0$. Then the formula for $q(x, F)$ takes the convenient loglinear form

$$\ln q(x, F) = \ln u(x) - \ln \left(\sum_{y \in F} u(y) \right) = \alpha + \beta U(x)$$

where the normalizing constant α is chosen to ensure that $\sum_{x \in F} q(x, F) = 1$. Obviously, this expression for $\ln q(x, F)$ is invariant under transformations taking the form $\tilde{U}(x) \equiv \gamma + U(x)$ for an arbitrary constant γ. A harmless normalization should be to choose utility units so that $\beta = 1$. In which case, whenever $x, y \in F \in \mathcal{F}$, the utility difference $U(x) - U(y)$ becomes equal to the logarithmic choice probability ratio $\ln[q(x, F)/q(y, F)]$.

7.2 Logit Equilibrium

Consider the normal form game $G = \langle N, S^N, v^N \rangle$, as in Section 6.1. For each player $i \in N$, assume that the multinomial logit version of Luce's model applies directly to the choice of strategy $s_i \in S_i$. Specifically, assume that there is a stochastic utility function of the form $f_i(s_i, \pi_i) = \exp[\beta_i V_i(s_i, \pi_i)]$, for some positive constant β_i. Then each player $i \in N$ has a *logit response function* $\pi_i \mapsto p_i(\pi_i)(\cdot)$ mapping $\Delta(S_{-i})$ to $\Delta(S_i)$ which satisfies

$$\ln[p_i(\pi_i)(s_i)] = \beta_i V_i(s_i, \pi_i) - \rho_i(\pi_i)$$

for all $\pi_i \in \Delta(S_{-i})$ and all $s_i \in S_i$, where the normalizing constant $\rho_i(\pi_i)$ is defined as the weighted exponential mean $\ln\left\{\sum_{s_i \in S_i} \exp[\beta_i V_i(s_i, \pi_i)]\right\}$.

Following McKelvey and Palfrey (1995), a *logit equilibrium* is defined as a profile $\bar{\mu}^N \in \prod_{i \in N} \Delta(S_i)$ of independent mixed strategies satisfying $\bar{\mu}_i(s_i) = p_i(\bar{\pi}_i)(s_i)$ for each player $i \in N$ and each strategy $s_i \in S_i$, where $\bar{\pi}_i = \bar{\mu}^{N\setminus\{i\}} = \prod_{h \in N\setminus\{i\}} \bar{\mu}_h$. In fact, such an equilibrium must be a fixed point of the mapping $p : D \to D$ defined on the domain $D := \prod_{i \in N} \Delta(S_i)$ by $p(\mu^N)(s^N) = \langle p_i(\mu^{N\setminus\{i\}})(s_i) \rangle_{s_i \in S_i}$. Note that this mapping, and the associated set of logit equilibria, are invariant under all increasing affine transformations of the form $\tilde{v}_i(s^N) \equiv \alpha_i(s_{-i}) + \delta_i v_i(s^N)$ provided that we replace each β_i with $\tilde{\beta}_i := \beta_i/\rho_i$. Indeed, one allowable transformation makes each $\tilde{\beta}_i = 1$, in which case each transformed utility difference satisfies

$$v_i(s_i, \bar{s}_{-i}) - v_i(s_i', \bar{s}_{-i}) = \ln[p_i(1_{\bar{s}_{-i}})(s_i)/p_i(1_{\bar{s}_{-i}})(s_i')]$$

where $1_{\bar{s}_{-i}}$ denotes the degenerate lottery that attaches probability 1 to the strategy profile $\bar{s}_{-i}$. Once again, utility differences become equal to logarithmic probability ratios.

8 Evolutionary Stability

8.1 Replicator Dynamics in Continuous Time

Let $G = \langle N, S^N, v^N \rangle$ be a game with cardinal utility, as defined in Section 6.1. Suppose that each $i \in N$ represents an entire population of players, rather than a single player. Suppose too that there are large and equal numbers of players in each population. All players are matched randomly in groups of size $\#N$, with one player from each population. The matching occurs repeatedly over time, and independently between time periods. At each moment of time every matched group of $\#N$ players plays the game G.

Among each population i, each strategy $s_i \in S_i$ corresponds to a player type. The proportion $\sigma_i(s_i)$ of players of each such type within the population i evolves over time. Assuming suitable random draws, each player in population i encounters a probability distribution $\pi_i \in \Delta(S_{-i})$ over other players' type profiles $s_{-i} \in S_{-i}$ that is given by

$$\pi_i(s_{-i}) = \bar{\pi}_i(\langle s_j \rangle_{j \in N \setminus \{i\}}) = \prod_{j \in N \setminus \{i\}} \sigma_j(s_j)$$

The expected payoff $V_i(s_i, \pi_i)$ experienced by any player of type s_i in population i is interpreted as a measure of (relative) "biological fitness". It is assumed that the rate of replication of that type of player depends on the difference between that measure of fitness and the average fitness $\mathbb{E}_{\sigma_i} V_i(\cdot, \pi_i)$ over the whole population i. It is usual to work in continuous time and to treat the dynamic process as deterministic because it is assumed that the populations are sufficiently large to eliminate any randomness.[7] Thus, one is led to study a *replicator dynamic process* in the form of simultaneous differential equations which determine the proportional net rate of growth $\hat{\sigma}_i(s_i) := \frac{d}{dt} \ln \sigma_i(s_i)$ of each type of player in each population.

8.2 Standard Replicator Dynamics

Following the ideas of Taylor and Jonker (1978) and Taylor (1979), the *standard replicator dynamics* (Weibull, 1995) occur when the differential equations imply that the proportional rate of growth $\hat{\sigma}_i(s_i)$ equals the measure of *excess fitness* defined by

$$E_i(s_i; \sigma_i, \pi_i) := V_i(s_i, \pi_i) - \mathbb{E}_{\sigma_i} V_i(\cdot, \pi_i)$$

for each $i \in N$ and each $s_i \in S_i$. In this case, consider affine transformations which take each player's payoff function from v_i to $\tilde{v}_i(s^N) \equiv \alpha_i(s_{-i}) + \delta_i v_i(s^N)$, where the multiplicative constants δ_i are all positive. This multiplies by δ_i each excess fitness function $E_i(s_i; \sigma_i, \pi_i)$, and so the transformed rates of population growth. Thus, these utility transformations in general speed up or slow down the replicator dynamics within each population. When $\delta_i = \delta$, independent of i, all rates adjust proportionately, and it is really just like measuring time in a different unit. Generally, however, invariance of the replicator dynamics requires all the affine transformations to be translations of the form $\tilde{v}_i(s^N) \equiv \alpha_i(s_{-i}) + v_i(s^N)$, with each $\delta_i = 1$, in effect. This is entirely appropriate because each utility difference

[7] See Boylan (1992) and Duffie and Sun (2004) for a discussion of this.

$v_i(s_i, \bar{s}_{-i}) - v_i(s_i', \bar{s}_{-i})$ equals the difference $E_i(s_i; \sigma_i, 1_{\bar{s}_{-i}}) - E_i(s_i'; \sigma_i, 1_{\bar{s}_{-i}})$ in excess fitness, which is independent of σ_i, and so equals the difference $\hat{\sigma}_i(s_i) - \hat{\sigma}_i(s_i')$ in proportional rates of growth.

8.3 Adjusted Replicator Dynamics

Weibull (1995) also presents a second form of *adjusted replicator dynamics*, based on Maynard Smith (1982). The proportional rates of growth become

$$\hat{\sigma}_i(s_i) = \frac{E_i(s_i; \sigma_i, \pi_i)}{\mathbb{E}_{\sigma_i} V_i(\cdot, \pi_i)} = \frac{V_i(s_i, \pi_i)}{\mathbb{E}_{\sigma_i} V_i(\cdot, \pi_i)} - 1$$

for each $i \in N$ and each $s_i \in S_i$. Then the above affine transformations have no effect on rates of population growth in the case when $\tilde{v}_i(s^N) \equiv \delta_i v_i(s^N)$, for arbitrary positive constants δ_i that can differ between populations. Thus, different utility functions are determined up to non-comparable ratio scales. Indeed, each utility ratio $v_i(s_i, \bar{s}_{-i})/v_i(s_i', \bar{s}_{-i})$ equals the excess fitness ratio $E_i(s_i; \sigma_i, 1_{\bar{s}_{-i}})/E_i(s_i'; \sigma_i, 1_{\bar{s}_{-i}})$, which is independent of σ_i, and so equals the ratio $\hat{\sigma}_i(s_i)/\hat{\sigma}_i(s_i')$ of the proportional rates of growth.

9 Consequentialist Foundations

9.1 Consequentialist Game Forms

Let X denote a fixed domain of possible consequences. A *consequentialist game form* is a triple $\Gamma = \langle N, S^N, \gamma \rangle$ where:

(i) N is a finite set of *players*;

(ii) each player $i \in N$ has a *strategy set* S_i, and $S^N = \prod_{i \in N} S_i$ is the set of *strategy profiles*;

(iii) there is an *outcome function* $\gamma : S^N \to X$ which specifies what consequence results from each strategy profile in the domain S^N.

Consider any fixed profile w^N of individual utility functions $w_i : X \to \mathbb{R}$ defined on the consequence domain X. Given any consequentialist game form Γ, there is a unique corresponding game $G^\Gamma(w^N) = \langle N, S^N, u^N \rangle$ with $u_i(s^N) \equiv w_i(\gamma(s^N))$ for all $s^N \in S^N$ and all $i \in N$. There is also a best reply correspondence

$$\bar{s}_{-i} \mapsto\!\!\!\to B_i^\Gamma(\bar{s}_{-i}; w_i) := \arg\max_{s_i \in S_i} w_i(\gamma(s_i, \bar{s}_{-i}))$$

and a (possibly empty) pure strategy Nash equilibrium set $E^\Gamma(w^N)$.

9.2 Ordinal Invariance

An obvious invariance property is that $B_i^\Gamma(\bar{s}_{-i}; w_i) \equiv B_i^\Gamma(\bar{s}_{-i}; \tilde{w}_i)$ for all possible Γ, which is true if and only if w_i and $\tilde{w}_i$ are ordinally equivalent functions on the domain X, for each $i \in N$. Similarly, all the other pure strategy solution concepts mentioned in Section 5.2.4, especially the pure strategy Nash equilibrium set $E^\Gamma(w^N)$, are preserved for all possible Γ if and only if the two profiles w^N and $\tilde{w}^N$ are ordinally equivalent. In this sense, we have reverted to the usual form of ONC invariance, rather than the SCONC invariance property that applies when just one game is being considered. This is one reason why the theory set out in Hammond (2004), for instance, does consider the whole class of consequentialist game forms.

9.3 Cardinal Invariance

A similar invariance concept applies when each player i's strategy set S_i is replaced by $\Delta(S_i)$, the set of mixed strategies, and the outcome function $\gamma : S^N \to X$ is replaced by a lottery outcome function $\gamma : \Delta(S^N) \to \Delta(X)$. Then all the players' best reply correspondences are preserved in all consequentialist game forms Γ if and only if the two profiles w^N and $\tilde{w}^N$ are cardinally equivalent. Similarly for any other solution concepts that depend only on the players' belief contingent preference orderings $R_i(\pi_i)$.

10 Concluding Remarks

Traditionally, game theorists have contented themselves with specifying a single numerical payoff function for each player. They do so without any consideration of the units in which utility is measured, or what alternative profiles of payoff functions can be regarded as equivalent. This paper will have succeeded if it leaves the reader with the impression that considering such measurement issues can considerably enrich our understanding of the decision-theoretic foundations of game theory. A useful by-product is identifying which games can be treated as equivalent to especially simple games, such as two-person zero-sum games, or team games.

Finally, it is pointed out that the usual utility concepts in single-person decision theory can be derived by considering different players' objectives in the whole class of consequentialist game forms, rather than just in one particular game.

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