Social Welfare Functionals on Restricted Domains and in Economic Environments

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Abstract: Arrow's "impossibility" and similar classical theorems are usually proved for an unrestricted domain of preference profiles. Recent work extends Arrow's theorem to various restricted but "saturating" domains of privately oriented, continuous, (strictly) convex, and (strictly) monotone "economic preferences" for private and/or public goods. For strongly saturating domains of more general utility profiles, this paper provides similar extensions of Wilson's theorem and of the strong and weak "welfarism" results due to d'Aspremont and Gevers and to Roberts. Hence, for social welfare functionals with or without interpersonal comparisons of utility, most previous classification results in social choice theory apply equally to strongly saturating economic domains.

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SWFLs in Economic Domains

1. Introduction and Outline

As far as the welfare economist is concerned, the main task of social choice theory is to aggregate measures of welfare for each individual into a Bergson–Samuelson social welfare function. Ideally, such aggregation should satisfy the Pareto condition (P) and some appropriate form of the independence (of irrelevant alternatives) condition (I). It has also been usual to impose condition (U), requiring that there be an unrestricted domain of individual preference profiles. Then, without interpersonal comparisons, Arrow's well known impossibility theorem shows that this aggregation requires a dictatorship. Sen (1970) extended Arrow's result to allow cardinal utility functions. In some sense, interpersonal comparisons are even needed in order to choose the dictator. Also, Wilson's (1972) interesting generalization of Arrow's theorem shows that, when condition (P) is weakened to a particular "weak non-imposition" condition (WNI), then unless the social welfare functional always leads to universal social indifference, there must be either a dictator or an inverse dictator.[1]

Interpersonal comparisons allow a much richer range of possibilities. Nevertheless, suppose that a social welfare functional (SWFL) satisfies conditions (U) and (I), both revised in an obvious way as stated in Hammond (1976, 1991), d'Aspremont and Gevers (1977) and Roberts (1980) so that the new conditions apply to profiles of utility functions which may be interpersonally comparable. Suppose too that the Pareto indifference condition (P^0) is satisfied. Then the important "strong neutrality" result due to d'Aspremont and Gevers (1977, Lemmas 2 and 3), Sen (1977) and d'Aspremont (1985, Theorem 2.3, p. 34) implies the existence of a "welfarist" preference ordering. This ordering is defined on the finite dimensional Euclidean space of vectors of utility levels and represents the social welfare functional in an obvious way. Alternatively, provided that both condition (P) and a weak form of continuity are satisfied, there is another important "weak neutrality" result due to Roberts (1980, Theorem 1, p. 428). Subject to a minor correction discussed in Hammond (1999) and in Section 3 below, this result shows that the social preference ordering has

[1] Wilson (1972) also explores the implications of abandoning even condition (WNI) and retaining only conditions (U) and (I). It would be fairly easy but not especially enlightening to extend the results of this paper for saturating domains in the same way.

a one-way "weak welfarist" representation by a continuous and weakly monotonic real-valued function of individual utility levels — that is, whenever this "Roberts social welfare function" (or RSWF) increases, the social state must be preferred.

In their original forms, all these theorems required there to be an unrestricted domain of individual preference orderings or utility functions, respectively. Yet welfare economics typically makes use of a Paretian Bergson–Samuelson social welfare function defined on a space of economic allocations. Then it is natural to consider only "economically relevant" preferences or utility functions for economic allocations. In fact, it is usual to assume that each individual's preferences are "privately oriented" in the sense of being indifferent to changes in other individuals' allocations of private goods. And to assume that each individual's preferences are at least weakly monotone as regards combinations of their own private goods with public goods. In the extreme case when there is no public good and only one private good, as in pure "cake division" problems, this implies that there must be a unique profile of individual preferences.

Despite this evident violation of condition (U), important later work has extended Arrow's impossibility theorem to different restricted domains of economically relevant preferences. See, for example, Kalai, Muller and Satterthwaite (1979), Gibbard, Hylland and Weymark (1987), Donaldson and Weymark (1988), Bordes and Le Breton (1989, 1990a, b), Campbell (1989b), Redekop (1990, 1993a, b, 1995), Le Breton and Weymark (1996), and Le Breton (1997). In particular, various kinds of "saturating" domain are rich enough for Arrow's theorem to hold.

The main purpose of this paper is to find similar extensions of the fundamental neutrality or welfarism theorems due to d'Aspremont and Gevers and to Roberts.[2] We shall also extend Wilson's theorem in the same direction, following the work of Bordes and Le Breton (1989) for the case of an economic domain with only private goods — see also Campbell (1989a).[3] These three extensions are more straightforward, however, if one first considers a more general, restricted, but "strongly saturating" domain of individual utility function profiles, not necessarily just a domain that arises in familiar economic environments.

[2] Recently, Weymark (1998) has independently proved the corresponding extension of d'Aspremont and Gevers' strong neutrality result.

[3] Bordes and Le Breton (1990b) also extend Wilson's theorem when the feasible set consists of "assignments, matchings, and pairings," but we do not consider this case explicitly.

One way to prove our results would be to check that each step of the original proofs by Arrow, Wilson, d'Aspremont and Gevers, and Roberts remains valid even in the relevant restricted domain. Our approach, by contrast, follows the technique pioneered by Kalai, Muller and Satterthwaite (1979). This involves applying the standard results for unrestricted domains locally to "free triples." The global result is then established by connecting "non-trivial" pairs of social states through chains of overlapping free triples, and invoking the independence axiom; special arguments are also needed to treat trivial pairs which, by definition, cannot be included in any free triple.

The paper is organized as follows. In section 2, we present the basic framework and definitions, followed in section 3 by the three main theorems. Section 4 contains proofs. Finally, section 5 considers adaptations of our results to treat three different economic environments: the first has only public goods and a common domain of individual preferences; the second has only private goods and individual preferences which depend only on their own consumption; and the third allows both private and public goods. The only previous results for this important third case appear in Bordes and Le Breton's (1990a) presentation of Arrow's theorem.

A brief appendix sets out a lemma, based on arguments due to Sen (1970) and to d'Aspremont and Gevers (1977), showing that cardinal non-comparability of individuals' utility functions implies ordinal non-comparability when conditions (U) and (I) are satisfied.

2. Definitions and Notation

Suppose that there is a society described by a finite set N of individuals, with typical member i. Let Z be the *underlying set* of all possible social states. Given any $S \subset Z$, let $\mathcal{U}(S)$ and $\mathcal{R}(S)$ respectively denote the domain of all possible utility functions $U : S \to \mathbb{R}$ and all possible (reflexive, complete and transitive) preference orderings R on S.

For each individual $i \in N$ and each $S \subset Z$, let $\mathcal{U}_i(S)$ and $\mathcal{R}_i(S)$ be copies of $\mathcal{U}(S)$ and $\mathcal{R}(S)$ respectively. The two Cartesian product sets $\mathcal{U}^N(S) := \prod_{i \in N} \mathcal{U}_i(S)$ and $\mathcal{R}^N(S) := \prod_{i \in N} \mathcal{R}_i(S)$ consist of all possible *utility profiles* and *preference profiles* respectively.

Given any utility function $U \in \mathcal{U}(Z)$, let $\psi(U) \in \mathcal{R}(Z)$ denote the corresponding preference ordering represented by U. Given any utility profile $\mathbf{U}^N \in \mathcal{U}^N(Z)$, let $\psi^N(\mathbf{U}^N) \in \mathcal{R}^N(Z)$ denote the associated preference profile.

Let $\mathcal{D}^N \subset \mathcal{U}^N(Z)$ be the restricted *domain* of allowable utility profiles. Then $\psi^N(\mathcal{D}^N)$ is the associated restricted domain of allowable preference profiles.

For each individual $i \in N$, let $\mathcal{D}_i := \text{proj}_i\, \mathcal{D}^N$ and $\mathcal{R}_i := \psi_i(\mathcal{D}_i)$ denote the associated domains of i's allowable utility functions and preference orderings, respectively.

Following Sen (1970, 1977), assume that there is a *social welfare functional* (or SWFL) $F : \mathcal{D}^N \to \mathcal{R}(Z)$ mapping each allowable utility profile $\mathbf{U}^N \in \mathcal{D}^N$ into a *social welfare ordering* $R = F(\mathbf{U}^N)$ on the underlying set Z. Let $P(\mathbf{U}^N)$ and $I(\mathbf{U}^N)$ respectively denote the corresponding strict preference and indifference relations; in accordance with this notation, $R(\mathbf{U}^N)$ will sometimes be used to denote $F(\mathbf{U}^N)$.

It will be assumed throughout that the SWFL F satisfies the standard independence condition (I). This requires that whenever $S \subset Z$ and the two utility function profiles $\mathbf{U}^N, \bar{\mathbf{U}}^N \in \mathcal{D}^N$ satisfy $\mathbf{U}^N(x) = \bar{\mathbf{U}}^N(x)$ for all $x \in S$, then for all $a, b \in S$ the associated social welfare orderings $R = F(\mathbf{U}^N)$ and $\bar{R} = F(\bar{\mathbf{U}}^N)$ should satisfy $a\ R\ b \iff a\ \bar{R}\ b$.

Arrow's impossibility and Roberts' weak neutrality theorems also rely on the Pareto condition (P), which requires that whenever $a, b \in Z$ and $\mathbf{U}^N \in \mathcal{D}^N$ satisfy $\mathbf{U}^N(a) \gg \mathbf{U}^N(b)$, then $a\ P(\mathbf{U}^N)\ b$.[4] The strong neutrality result of d'Aspremont and Gevers (1977) makes use of the alternative *Pareto indifference* condition (P^0) requiring that $a\ I(\mathbf{U}^N)\ b$ whenever $a, b \in Z$ and $\mathbf{U}^N(a) = \mathbf{U}^N(b)$. Also, say that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies the *strict Pareto condition* (P*) provided that, whenever $a, b \in Z$, then $\mathbf{U}^N(a) \geqq \mathbf{U}^N(b)$ implies $a\ R(\mathbf{U}^N)\ b$, while $\mathbf{U}^N(a) > \mathbf{U}^N(b)$ implies $a\ P(\mathbf{U}^N)\ b$.

Given any individual $i \in N$ and subset $S \subset Z$, let $\psi(\mathcal{D}_i)_{|S}$ denote the set of restrictions to S of preference orderings $\psi(U)$ for some $U \in \mathcal{D}_i$. Then $S \subset Z$ is said to be *individually trivial* (or *trivial relative to* $\mathcal{D}_i$) if there exists $i \in N$ such that $\#\psi(\mathcal{D}_i)_{|S} = 1$. Hence, triviality requires there to be some individual with a unique preference relation $R_{i|S} = \psi(U_i)_{|S}$ on S corresponding to every allowable utility function $U_i \in \mathcal{D}_i$. In other words, the restriction to S of every $U_i \in \mathcal{D}_i$ must represent the same restricted preference ordering $R_{i|S}$. Also, $S \subset Z$ is said to be *trivial* (or *trivial relative to* $\mathcal{D}^N$) if there exists $i \in N$ such

[4] We use the following notation for vector inequalities. Given any pair $\mathbf{u}^N = \langle u_i \rangle_{i \in N}$ and $\mathbf{v}^N = \langle v_i \rangle_{i \in N}$, define $\mathbf{u}^N \gg \mathbf{v}^N \iff u_i > v_i$ (all $i \in N$), $\mathbf{u}^N \geqq \mathbf{v}^N \iff u_i \geq v_i$ (all $i \in N$), and $\mathbf{u}^N > \mathbf{v}^N \iff [\mathbf{u}^N \geqq \mathbf{v}^N$ but $\mathbf{u}^N \neq \mathbf{v}^N]$.

that $\#\psi(\mathcal{D}_i)_{|S} = 1$. Equivalently, $S \subset Z$ is *non-trivial* (relative to $\mathcal{D}^N$) if $\#\psi(\mathcal{D}_i)_{|S} \geq 2$ for all $i \in N$.

Given any individual $i \in N$ and any subset $S \subset Z$, let $\mathcal{D}_{i|S}$ denote the set of restrictions to S of utility functions $U_i \in \mathcal{D}_i$. Then $S \subset Z$ is said to be *individually utility free* (or *utility free relative to* $\mathcal{D}_i$) if $\mathcal{D}_{i|S} = \mathcal{U}(S)$. This requires every possible utility function on the set S to have an extension to some allowable function on Z that lies in the domain $\mathcal{D}_i$. Equivalently, the domain condition $\mathbf{U}_i \in \mathcal{D}_i$ must leave $\mathbf{U}_i$ completely unrestricted on S. The set $S \subset Z$ is said to be *utility free* (relative to $\mathcal{D}^N$) if $\mathcal{D}^N_{|S} = \mathcal{U}^N(S)$.

Two non-trivial pairs $\{ x, y \}$ and $\{ a, b \}$ (with $\{ x, y \} \neq \{ a, b \}$) are said to be *individually connected* in $\mathcal{D}_i$ (by individually utility free triples) if for some $r = 3, 4, \ldots$ there exists a chain $\{ z_k \in Z \mid k = 1, 2, \ldots r \}$ made up of $r - 2$ overlapping individually utility free triples $T_k = \{ z_{k-1}, z_k, z_{k+1} \}$ which connects $\{ z_1, z_2 \} = \{ x, y \}$ at the beginning to $\{ z_{r-1}, z_r \} = \{ a, b \}$ at the end. The same two pairs are said to be *connected* in $\mathcal{D}^N$ if there exists a similar chain $\{ z_k \in Z \mid k = 1, 2, \ldots r \}$ of utility free triples relative to $\mathcal{D}^N$. Obviously, if either $\{ x, y \}$ or $\{ a, b \}$ is a trivial pair, there is no way of connecting $\{ x, y \}$ to $\{ a, b \}$ by utility free triples. For this reason, trivial pairs need separate treatment.

The domain $\mathcal{D}^N$ of preference profiles is *saturating* if Z includes at least two non-trivial pairs, and if every non-trivial pair in Z is connected to every other non-trivial pair through utility free triples. In particular, the domain $\mathcal{D}^N$ can be saturating only if Z contains at least one free triple. Note that the two non-trivial pairs must be different but may intersect. The domain $\mathcal{D}^N$ may be saturating, therefore, even if Z has only three members. Indeed, $\mathcal{D}^N$ will be saturating if Z is a utility free triple. This is important because it shows that, when $\#Z \geq 3$, our main Theorems 1–3 for appropriate kinds of saturating domain have as corollaries the usual results for an unrestricted domain of utility functions.

As Result 3 in Section 4 below shows, the above saturating domain condition ensures that our theorems do apply to all non-trivial pairs. To include trivial pairs as well, we follow the usual practice of introducing stronger forms of saturation.

The domain $\mathcal{D}^N$ is *strongly saturating* if it is saturating and also *separable* (cf. Kalai and Ritz, 1980) in the sense that, for every trivial pair $\{ a, b \} \subset Z$ and utility profile $\mathbf{U}^N \in \mathcal{D}^N$, there exists $c \in Z$ for which $\{ a, c \}$ and $\{ b, c \}$ are non-trivial pairs, while for any $\lambda \in [0, 1]$

there exists a profile $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ satisfying:

$$\bar{\mathbf{U}}^N(a) = \mathbf{U}^N(a); \quad \bar{\mathbf{U}}^N(b) = \mathbf{U}^N(b); \quad \bar{\mathbf{U}}^N(c) = (1-\lambda)\mathbf{U}^N(a) + \lambda\mathbf{U}^N(b).$$

In fact, this last condition can be somewhat weakened for two of our results. Specifically, the domain $\mathcal{D}^N$ is *less strongly saturating* if it is saturating, and also the above equations are required to hold only for $\lambda = 0$, implying that $\bar{\mathbf{U}}^N(a) = \bar{\mathbf{U}}^N(c) = \mathbf{U}^N(a)$ and $\bar{\mathbf{U}}^N(b) = \mathbf{U}^N(b)$ for some $c \in Z$ such that $\{a, c\}$ and $\{b, c\}$ are both non-trivial.

Finally, the domain $\mathcal{D}^N$ is *ordinally strongly saturating* if it is saturating and also, for every trivial pair $\{a, b\} \subset Z$, individual $i \in N$, and utility profile $\mathbf{U}^N \in \mathcal{D}^N$ that satisfies $U_i(a) > U_i(b)$, there exist $c \in Z$ and an alternative profile $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ such that $\{a, c\}$ and $\{b, c\}$ are non-trivial pairs, while $\bar{\mathbf{U}}^N(a) = \mathbf{U}^N(a)$, $\bar{\mathbf{U}}^N(b) = \mathbf{U}^N(b)$, and $\bar{U}_i(a) > \bar{U}_i(c) > \bar{U}_i(b)$. Clearly, this is a different and major weakening of the strongly saturating domain condition.

3. Statement of the Main Theorems

3.1. *Arrow's and Wilson's Theorems*

The three main theorems set out in this section will be proved in the subsequent Section 4 via a series of intermediate Results.

Say that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies the *weak non-imposition* condition (WNI) provided that, for every utility free set $S \subset Z$ and every $a, b \in S$, there exists $\mathbf{U}^N \in \mathcal{D}^N$ such that $a\ R(\mathbf{U}^N)\ b$. This condition weakens Wilson's (1972) assumption by requiring non-imposition to hold only on utility free sets. This weakening seems appropriate when there is a restricted domain of preferences and utility profiles. Obviously, condition (WNI) is a considerable weakening of the Pareto condition (P).

Following d'Aspremont and Gevers (1977) and Roberts (1980), say that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies the *ordinal non-comparability* condition (ONC) provided that $\mathbf{U}^N \in \mathcal{D}^N$, $\bar{\mathbf{U}}^N \in \mathcal{U}^N(Z)$ and $\psi^N(\bar{\mathbf{U}}^N) = \psi^N(\mathbf{U}^N)$ together imply that $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ and $F(\bar{\mathbf{U}}^N) = F(\mathbf{U}^N)$. Similarly, say that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies the *cardinal non-comparability* condition (CNC) provided that, whenever $\mathbf{U}^N \in \mathcal{D}^N$ and $\bar{\mathbf{U}}^N \in \mathcal{U}^N(Z)$ are such that there exist additive constants α_i and multiplicative constants $\rho_i > 0$ for which

$\bar{U}_i(z) \equiv \alpha_i + \rho_i U_i(z)$ throughout Z, then $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ and $F(\bar{\mathbf{U}}^N) = F(\mathbf{U}^N)$. As pointed out in the appendix, Sen's proof of Arrow's theorem for cardinal utility functions (see Sen, 1970, Theorem 8*2, pp. 129–30) shows that any SWFL satisfying both (I) and (CNC) must also satisfy (ONC). Also, as is well known, any SWFL satisfying (ONC) is effectively an Arrow social welfare function.

Given the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ and the set $S \subset Z$, say that there is *universal indifference for* S provided that, for all $a, b \in S$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has $a\ I(\mathbf{U}^N)\ b$ always. Similarly, say that $d \in N$ is a *dictator for* S if, for all $a, b \in S$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has $a\ P(\mathbf{U}^N)\ b$ whenever $U_d(a) > U_d(b)$, and that $d \in N$ is an *inverse dictator for* S if instead $a\ P(\mathbf{U}^N)\ b$ whenever $U_d(a) < U_d(b)$. When $S = Z$, simply say that there is *universal indifference*, or that d is a *dictator* or *inverse dictator*, without referring to Z explicitly.

Using condition (P) instead of (WNI), "Arrovian theorems" concerning the existence of a dictator have been proved for particular economic environments by Kalai, Muller and Satterthwaite (1979), followed by Bordes and Le Breton (1989, 1990a, b). The following result represents a minor but useful generalization:

Theorem 1 (Wilson). *Suppose that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies conditions (I), (WNI), and (CNC) on an ordinally strongly saturating domain $\mathcal{D}^N$. Then, unless there is universal indifference, there exists either a dictator or an inverse dictator.*

The familiar anonymity condition (A) requires all individuals' preferences or utilities to be treated symmetrically; in particular, it excludes both a dictatorship and also an inverse dictatorship. The familiar neutrality condition (N), on the other hand, requires a symmetric treatment of all social states. More interesting in the context of this paper is the *restricted neutrality* condition (RN) requiring that (N) be satisfied by the restriction of the SWFL to any utility free set. Specifically, given any utility free set S, whenever the social states $a, b, x, y \in S$ and the two utility function profiles $\mathbf{U}^N, \bar{\mathbf{U}}^N \in \mathcal{D}^N$ satisfy $\mathbf{U}^N(a) = \bar{\mathbf{U}}^N(x)$ as well as $\mathbf{U}^N(b) = \bar{\mathbf{U}}^N(y)$, then condition (RN) requires that $a\ R(\mathbf{U}^N)\ b$ iff $x\ R(\bar{\mathbf{U}}^N)\ y$.

A corollary of Theorem 1 is the following result, which Hansson (1969) originally proved for an unrestricted domain:

Corollary (Hansson). *Suppose that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies conditions (I), (A), (RN) and (CNC) on an ordinally strongly saturating domain $\mathcal{D}^N$. Then there is universal indifference.*[5]

Proof: It is enough to prove that condition (RN) implies (WNI). So suppose that $a, b \in S$ where $S \subset Z$ is a utility free set. Suppose that $a\ P(\mathbf{U}^N)\ b$ for some $\mathbf{U}^N \in \mathcal{D}^N$. Then condition (RN) implies that $b\ P(\bar{\mathbf{U}}^N)\ a$ when $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ satisfies $\bar{\mathbf{U}}^N(a) = \mathbf{U}^N(b)$, $\bar{\mathbf{U}}^N(b) = \mathbf{U}^N(a)$, and $\bar{\mathbf{U}}^N(x) = \mathbf{U}^N(x)$ for all $x \in S \setminus \{a, b\}$. This confirms that (WNI) holds. ▮

3.2. Strong Welfarism

Given the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$, say that the complete, reflexive and transitive binary relation $\succsim$ on $\mathbb{R}^N$ is a *welfarist ordering* if, for all $a, b \in Z$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has

$$\mathbf{U}^N(a) \succsim \mathbf{U}^N(b) \iff a\ R(\mathbf{U}^N)\ b$$

Similarly, say that $\succsim$ is a *restricted welfarist ordering for S* if the same property holds for all $a, b \in S$. Adding the extra Pareto indifference condition (P^0) allows one to prove the following version of an important result which d'Aspremont and Gevers (1977, Lemmas 2 and 3) first demonstrated.[6]

Theorem 2 (Strong Welfarism). *Suppose that the SWFL $F : \mathcal{D}^N \to \mathcal{R}(Z)$ satisfies conditions (I) and (P^0) on a less strongly saturating domain $\mathcal{D}^N$. Then there exists a welfarist ordering $\succsim$.*

This theorem can be used to show that many results such as those surveyed by d'Aspremont (1985) apply also to a less strongly saturating instead of an unrestricted domain of utility profiles — see also Weymark (1998), Mongin and d'Aspremont (1998), Bossert and Weymark (2004).

[5] If condition (RN) were replaced by the more demanding condition (N), then whenever $\mathcal{D}^N$ allows the existence of at least one utility free triple, the Corollary would obviously follow from Wilson's original version of Theorem 1 which holds for an unrestricted domain.

[6] Actually, instead of (P^0), d'Aspremont and Gevers assumed condition (P*) — which they called SP. However, it is easy to check that their proof of strong welfarism (on pp. 205–6) only uses conditions (U), (I) and (P^0), not (P*). Note that the proof in d'Aspremont and Gevers (1977), like that in Hammond (1979, p. 1129), is actually incomplete because not every possible case is treated. The proof in d'Aspremont (1985) is complete, but seems to require a minor correction: the option e should be chosen so that $b \neq e \neq d$, otherwise it may be impossible to construct U^4 and U^5 as required when $d = e$. Sen (1977) has a complete but less concise proof of strong welfarism.

3.3. Roberts' Weak Welfarism Result

In order to derive our third result, based on Roberts (1980, Theorem 1, p. 428), we make use of the following *pairwise continuity* condition (PC):[7]

For every $\mathbf{e} \gg \mathbf{0}$ *there exists* $\mathbf{e}' \gg \mathbf{0}$ *with the property that, whenever the pair* $x, y \in Z$ *and the profile* $\mathbf{U}^N \in \mathcal{D}^N$ *satisfy* $x\ P(\mathbf{U}^N)\ y$*, there is another profile* $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ *such that* $x\ P(\bar{\mathbf{U}}^N)\ y$*, while* $\bar{\mathbf{U}}^N(x) \ll \mathbf{U}^N(x) - \mathbf{e}'$ *and* $\bar{\mathbf{U}}^N(y) \gg \mathbf{U}^N(y) - \mathbf{e}$.

Say that the function $W : \mathbb{R}^N \to \mathbb{R}$ is a (one way) *Roberts social welfare function* (or RSWF) if W is continuous and weakly monotone,[8] while for all $a, b \in Z$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has

$$W(\mathbf{U}^N(a)) > W(\mathbf{U}^N(b)) \Longrightarrow a\ P(\mathbf{U}^N)\ b$$

In particular, W represents a *weak welfarist* preference ordering R^* on the space $\mathbb{R}^N$ whose corresponding strict preference relation P^* has the property that, for all $a, b \in Z$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has

$$W(\mathbf{U}^N(a)) > W(\mathbf{U}^N(b)) \iff \mathbf{U}^N(a)\ P^*\ \mathbf{U}^N(b) \Longrightarrow a\ P(\mathbf{U}^N)\ b$$

Similarly, given any set $S \subset Z$, say that $W_{|S} : \mathbb{R}^N \to \mathbb{R}$ is a *restricted* RSWF *for* S if $W_{|S}$ is continuous and weakly monotone, while for all $a, b \in S$ and $\mathbf{U}^N \in \mathcal{D}^N$, one has

$$W_{|S}(\mathbf{U}^N(a)) > W_{|S}(\mathbf{U}^N(b)) \Longrightarrow a\ P(\mathbf{U}^N)\ b$$

After these necessary preliminaries, the following result extends Roberts' weak welfarism theorem to strongly saturating domains in the same way as Theorems 1 and 2 respectively extend Wilson's theorem and the strong welfarism theorem due to d'Aspremont and Gevers.

Theorem 3 (Weak Welfarism). *Suppose that the SWFL* $F : \mathcal{D}^N \to \mathcal{R}(Z)$ *satisfies conditions (I), (P) and (PC) on a strongly saturating domain* $\mathcal{D}^N$*. Then there exists an RSWF.*

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[7] Actually, Roberts' theorem is incorrect as stated, but is easily corrected by using condition (PC) instead of weak continuity. See Hammond (1999) for details.

[8] Say that the function $W : \mathbb{R}^N \to \mathbb{R}$ is *weakly monotone* if, whenever $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$ with $\mathbf{u}^N \gg \mathbf{v}^N$, then $W(\mathbf{u}^N) > W(\mathbf{v}^N)$. When W is continuous, this implies that $W(\mathbf{u}^N) \geq W(\mathbf{v}^N)$ whenever $\mathbf{u}^N \geqq \mathbf{v}^N$.

It is worth noting that condition (PC) can be satisfied by utility function profiles satisfying five of the six comparability conditions summarized by Roberts (1980) — namely (ONC), (CNC), (OLC) (ordinal level comparability), (CUC) (cardinal unit comparability), and (CUC) (cardinal full comparability). But the sixth comparability condition — (CRS) (cardinal ratio scales) — violates condition (PC) unless the domain is restricted to positive-valued utilities, for which one can replace each $U_i(x)$ by $\log U_i(x)$, or to negative-valued utilities, for which one can replace each $U_i(x)$ by $-\log[-U_i(x)]$. Thus, Theorem 3 shows that, after correcting Roberts' Theorem by replacing condition (WC) with (PC), almost all his classification results apply to a strongly saturating as well as to an unrestricted domain of utility function profiles.

If (PC) is replaced by the more demanding condition (CNC), then Roberts (1980) shows that there must exist $i \in N$ such that $W(\mathbf{u}^N) = u_i$. This shows the logical connection between: (i) Theorem 1 when (WNI) is strengthened to (P); (ii) Theorem 3 when (PC) is strengthened to (CNC).

4. Proof of the Main Theorems

Throughout this section, let $F : \mathcal{D}^N \to \mathcal{R}(Z)$ be any SWFL satisfying condition (I) on the restricted domain $\mathcal{D}^N$.

RESULT 1. *Let $S \subset Z$ be any utility free set with $\#S \geq 3$. (a) If F satisfies conditions (WNI) and (CNC), then unless there is universal indifference for S, there exists either a dictator or an inverse dictator for S. (b) If F satisfies condition (P^0), then there exists a welfarist ordering for S. (c) If F satisfies conditions (P) and (PC), then there exists a corresponding (continuous and weakly monotone) restricted RSWF $W_{|S} : \mathbb{R}^N \to \mathbb{R}$ for S.*

PROOF: By condition (I), the restricted SWFL $F_{|S} : \mathcal{U}^N(S) \to \mathcal{R}(S)$ is well defined, and satisfies conditions (U) and (I) on the unrestricted domain $\mathcal{U}^N(S)$ of all utility profiles defined on the utility free set S.

(a) Because F satisfies conditions (I), (WNI) and (CNC), its restriction $F_{|S}$ evidently satisfies the same three conditions on the free set S. But then the appendix shows that $F_{|S}$ satisfies (ONC) on S. Hence, $F_{|S}$ is equivalent to an Arrow social welfare function satisfying conditions (U), (I) and (WNI) on the unrestricted domain $\mathcal{R}^N(S)$ of all preference profiles defined on the free set S. So Theorem 3 of Wilson (1972, p. 484) can be applied, which yields the stated result.

(b) If condition (P^0) is satisfied, then $F_{|S}$ also satisfies condition (P^0). So the strong neutrality theorem of d'Aspremont and Gevers can be applied to $F_{|S}$, which yields the stated result.

(c) If conditions (P) and (PC) are satisfied, then $F_{|S}$ also satisfies conditions (P) and (PC). So Roberts' theorem can be applied to $F_{|S}$, which yields the stated result. ∎

A preference ordering R^* on $\mathbb{R}^N$ is said to be *weakly monotone* if $\mathbf{u}^N \; P^* \; \mathbf{v}^N$ whenever $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$ with $\mathbf{u}^N \gg \mathbf{v}^N$.

CLAIM. *Suppose that R_1^* and R_2^* are continuous weakly monotonic preference orderings on $\mathbb{R}^N$. Unless $R_1^* = R_2^*$, there exist $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$ such that $\mathbf{u}^N \; P_1^* \; \mathbf{v}^N$ but $\mathbf{v}^N \; P_2^* \; \mathbf{u}^N$.*

PROOF: Suppose that $R_1^* \neq R_2^*$. After interchanging R_1^* and R_2^* if necessary, completeness of these two orderings implies that there exist $\mathbf{u}^N, \mathbf{w}^N \in \mathbb{R}^N$ such that $\mathbf{u}^N \; P_1^* \; \mathbf{w}^N$ but $\mathbf{w}^N \; R_2^* \; \mathbf{u}^N$. Then weak monotonicity and transitivity together imply that for all $\mathbf{e}^N \gg 0$ one has $\mathbf{w}^N + \mathbf{e}^N \; P_2^* \; \mathbf{w}^N$ and so $\mathbf{w}^N + \mathbf{e}^N \; P_2^* \; \mathbf{u}^N$. But continuity implies that $\mathbf{u}^N \; P_1^* \; \mathbf{w}^N + \mathbf{e}^N$ for all small enough $\mathbf{e}^N \gg 0$. Hence, the result is true for $\mathbf{v}^N := \mathbf{w}^N + \mathbf{e}^N$ when $\mathbf{e}^N \gg 0$ is small enough. ∎

RESULT 2. *Suppose that $T_1, T_2 \subset Z$ are two utility free triples whose intersection $S := T_1 \cap T_2$ is a pair with members a, b. (a) If conditions (WNI) and (CNC) are satisfied, then unless there is universal indifference on $T_1 \cup T_2$, there must exist either a common dictator or a common inverse dictator for both sets T_1 and T_2. (b) If condition (P^0) is satisfied, then there exists a common welfarist ordering for both T_1 and T_2. (c) If conditions (P) and (PC) are satisfied, then the two restricted RSWFs $W_{|T_1}, W_{|T_2} : \mathbb{R}^N \to \mathbb{R}$ represent a common ordering R^* on $\mathbb{R}^N$.*

PROOF: (a) If conditions (WNI) and (CNC) are satisfied, then Result 1 implies that, unless there is universal indifference for T_1, there exists either a dictator or an inverse dictator for T_1. And similarly for T_2.

First, if there is universal indifference for T_1, then $a \; I(\mathbf{U}^N) \; b$ for all $\mathbf{U}^N \in \mathcal{D}^N$. Because $a, b \in T_2$ and T_2 is a utility free triple, this excludes the possibility of there being either a dictator or an inverse dictator for T_2. Hence, there must be universal indifference for T_2 as well, and so for $T_1 \cup T_2$.

Second, if $d \in N$ is a dictator for T_1, then $U_d(a) > U_d(b)$ implies that $a \; P(\mathbf{U}^N) \; b$. Because $a, b \in T_2$ and T_2 is a utility free triple, the only possibility allowed by Result 1 is that d is also a dictator for T_2.

Third, if $d \in N$ is an inverse dictator for T_1, then $U_d(a) < U_d(b)$ implies that $a \; P(\mathbf{U}^N) \; b$. Because $a, b \in T_2$ and T_2 is a utility free triple, the only possibility allowed by Result 1 is that d is also an inverse dictator for T_2.

(b) If condition (P^0) is satisfied, then Result 1 implies that there exist two welfarist orderings $\succsim_1$ and $\succsim_2$ for T_1 and T_2 respectively. Because T_1 and T_2 are free triples, for any pair $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$ there must exist $\mathbf{U}^N \in \mathcal{D}^N$ such that $\mathbf{U}^N(a) = \mathbf{u}^N$ and $\mathbf{U}^N(b) = \mathbf{v}^N$. Then

$$\mathbf{u}^N \;\succsim_1\; \mathbf{v}^N \iff a \; R(\mathbf{U}^N) \; b \iff \mathbf{u}^N \;\succsim_2\; \mathbf{v}^N$$

This is true for all $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$, so $\succsim_1 = \succsim_2$.

(c) If conditions (P) and (PC) are satisfied, then Result 1 implies that there exist two continuous and weakly monotone orderings R_1^* and R_2^* on $\mathbb{R}^N$ which are represented by $W_{|T_1}$ and $W_{|T_2}$ respectively. Unless $R_1^* = R_2^*$, the previous Claim establishes that there exist $\mathbf{u}^N, \mathbf{v}^N \in \mathbb{R}^N$ such that $\mathbf{u}^N \; P_1^* \; \mathbf{v}^N$ and $\mathbf{v}^N \; P_2^* \; \mathbf{u}^N$. Because T_1 and T_2 are free triples, there exists $\mathbf{U}^N \in \mathcal{D}^N$ such that $\mathbf{U}^N(a) = \mathbf{u}^N$ and $\mathbf{U}^N(b) = \mathbf{v}^N$. By definition of the orderings R_1^* and R_2^*, then $\mathbf{u}^N \; P_1^* \; \mathbf{v}^N$ implies that $a \; P(\mathbf{U}^N) \; b$, whereas $\mathbf{v}^N \; P_2^* \; \mathbf{u}^N$ implies that $b \; P(\mathbf{U}^N) \; a$, a contradiction. So $R_1^* = R_2^*$. ▮

Result 3. *Suppose that the domain $\mathcal{D}^N$ is saturating. (a) If conditions (WNI) and (CNC) are satisfied, then unless there is universal indifference on every non-trivial pair, there must exist either a common dictator or a common inverse dictator for every non-trivial pair. (b) If condition (P^0) is satisfied, then there exists a common welfarist ordering for every non-trivial pair. (c) If conditions (P) and (PC) are satisfied, then there exists a common restricted RSWF $W : \mathbb{R}^N \to \mathbb{R}$ for every non-trivial pair.*

Proof: Because the domain $\mathcal{D}^N$ is saturating, there exist at least two non-trivial pairs $\{x, y\}$ and $\{x', y'\}$ (with $\{x, y\} \neq \{x', y'\}$). Moreover, because $\{x, y\}$ and $\{x', y'\}$ are connected by a chain of free triples, there exists $z \in Z$ such that $T_1 := \{x, y, z\}$ is a free triple. This implies that Result 1 applies to T_1, and also that $\{y, z\}$ is a non-trivial pair.

Let $\{a, b\} \subset Z$ be any non-trivial pair other than $\{x, y\}$. Because the domain $\mathcal{D}^N$ is saturating, there exists a chain of overlapping utility free triples $T_k := \{z_{k-1}, z_k, z_{k+1}\}$ ($k = 1$ to $r \geq 1$) such that $\{z_0, z_1, z_2\} = \{x, y, z\}$ and $\{z_r, z_{r+1}\} = \{a, b\}$.

(a) When conditions (WNI) and (CNC) are satisfied, Result 1 implies that for each triple T_k ($k = 1$ to r), unless there is universal indifference for T_k, there exists either a dictator or an inverse dictator d_k for T_k. By induction on k, Result 2 implies that unless there is universal indifference on every T_k, there is either one common dictator or one common inverse dictator for every T_k, including T_r. Hence, for every non-trivial pair

$\{a, b\}$, there must be either universal indifference, or the same dictator, or the same inverse dictator.

(b) If condition (P^0) is satisfied, then Result 1 implies that there exists a welfarist ordering for each triple T_k ($k = 1$ to r). By Result 2, it follows by induction on k that these welfarist orderings are all the same. So there must exist some common welfarist ordering $\succsim$ on $\mathbb{R}^N$ which applies to every non-trivial pair.

(c) If conditions (P) and (PC) are satisfied, then Result 1 implies that there exists a restricted RSWF $W_{|T_k} : \mathbb{R}^N \to \mathbb{R}$ for each triple T_k ($k = 1$ to r). By Result 2, it follows by induction on k that the restricted RSWFs $W_{|T_k}$ ($k = 1$ to r) all represent the same ordering on $\mathbb{R}^N$. Any one of these restricted RSWFs will serve as a common RSWF $W : \mathbb{R}^N \to \mathbb{R}$ which applies to every non-trivial pair. ∎

Result 4. *Let $\{a, b\} \subset Z$ be any trivial pair. (a) If the domain $\mathcal{D}^N$ is ordinally strongly saturating and conditions (WNI) and (CNC) are satisfied, then unless there is universal indifference for all pairs, both trivial and non-trivial, either the common dictator of Result 3 is also a dictator for $\{a, b\}$, or the common inverse dictator of Result 3 is also an inverse dictator for $\{a, b\}$. (b) If the domain $\mathcal{D}^N$ is less strongly saturating and condition (P^0) is satisfied, then the welfarist ordering $\succsim$ of Result 3 also applies to $\{a, b\}$. (c) If the domain $\mathcal{D}^N$ is strongly saturating and conditions (P) and (PC) are satisfied, then the RSWF $W : \mathbb{R}^N \to \mathbb{R}$ of Result 3 also applies to $\{a, b\}$.*

Proof: (a) Consider first the case when there is universal indifference for every non-trivial pair. Because the domain is ordinally strongly saturating, there exist $c \in Z$ such that $\{a, c\}$ and $\{b, c\}$ are non-trivial pairs. Then, for all $\mathbf{U}^N \in \mathcal{D}^N$, one must have $a\ I(\mathbf{U}^N)\ c$ and $c\ I(\mathbf{U}^N)\ b$. Now transitivity implies that $a\ I(\mathbf{U}^N)\ b$, so universal indifference extends to the trivial pair $\{a, b\}$. Moreover, this must be true for every trivial pair, so in this case there is universal indifference for all pairs, both trivial and non-trivial.

More generally, suppose that $\mathcal{D}^N$ is ordinally strongly saturating and that conditions (WNI) and (CNC) are satisfied. Then Result 3 implies that there is either a dictator or an inverse dictator $d \in N$ for all non-trivial pairs. Suppose that $U_d(a) > U_d(b)$. Ordinal strong saturation implies that there exist $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ and $c \in Z$ such that $\bar{U}_i(a) = U_i(a)$ and $\bar{U}_i(b) = U_i(b)$ for all $i \in N \setminus \{d\}$, while $U_d(a) = \bar{U}_d(a) > \bar{U}_d(c) > \bar{U}_d(b) = U_d(b)$.

In case there is a dictator $d \in N$ for all non-trivial pairs, it follows that $a\ P(\bar{\mathbf{U}}^N)\ c$ and $c\ P(\bar{\mathbf{U}}^N)\ b$. Because $P(\bar{\mathbf{U}}^N)$ is transitive, it follows that $a\ P(\bar{\mathbf{U}}^N)\ b$. But $\mathbf{U}^N{}_{|\{a,b\}} = \bar{\mathbf{U}}^N{}_{|\{a,b\}}$ and so, because of condition (I), $a\ P(\mathbf{U}^N)\ b$.

In case there is an inverse dictator $d \in N$ for all non-trivial pairs, a similar argument shows that $c\ P(\bar{\mathbf{U}}^N)\ a$ and $b\ P(\bar{\mathbf{U}}^N)\ c$, implying that $b\ P(\bar{\mathbf{U}}^N)\ a$ and so, by condition (I), that $b\ P(\mathbf{U}^N)\ a$.

(b) If condition (P^0) is satisfied and the domain $\mathcal{D}^N$ is less strongly saturating, there exist $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ and $c \in Z$ such that $\bar{\mathbf{U}}^N(c) = \bar{\mathbf{U}}^N(a) = \mathbf{U}^N(a)$ and $\bar{\mathbf{U}}^N(b) = \mathbf{U}^N(b)$. Let $\succsim$ be the welfarist ordering of Result 3, which applies to every non-trivial pair. In this case, our constructions imply that

$$\begin{aligned}
&\mathbf{U}^N(a) \succsim \mathbf{U}^N(b) \quad \Longrightarrow \quad \bar{\mathbf{U}}^N(a) = \bar{\mathbf{U}}^N(c) \succsim \bar{\mathbf{U}}^N(b) \\
\Longrightarrow \quad & a\ I(\bar{\mathbf{U}}^N)\ c\ R(\bar{\mathbf{U}}^N)\ b \quad \text{(using condition (P}^0\text{) and non-triviality of } \{\, b, c\,\}) \\
\Longrightarrow \quad & a\ R(\bar{\mathbf{U}}^N)\ b \quad \text{(because } R \text{ is transitive)} \quad \Longrightarrow \quad a\ R(\mathbf{U}^N)\ b \quad \text{(by condition (I))}
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbf{U}^N(b) \succ \mathbf{U}^N(a) \quad \Longrightarrow \quad \bar{\mathbf{U}}^N(b) \succ \bar{\mathbf{U}}^N(c) = \bar{\mathbf{U}}^N(a) \\
\Longrightarrow \quad & b\ P(\bar{\mathbf{U}}^N)\ c\ I(\bar{\mathbf{U}}^N)\ a \quad \text{(using condition (P}^0\text{) and non-triviality of } \{\, b, c\,\}) \\
\Longrightarrow \quad & b\ P(\bar{\mathbf{U}}^N)\ a \quad \text{(because } R \text{ is transitive)} \quad \Longrightarrow \quad b\ P(\mathbf{U}^N)\ a \quad \text{(by condition (I))}
\end{aligned}$$

Hence $\mathbf{U}^N(a) \succsim \mathbf{U}^N(b) \iff a\ R(\mathbf{U}^N)\ b$, implying that the same welfarist ordering $\succsim$ applies to the trivial pair $\{\, a, b\,\}$.

(c) Let $\mathbf{U}^N \in \mathcal{D}^N$ be any utility profile. Define $\mathbf{u}^N := \mathbf{U}^N(a)$ and $\mathbf{v}^N := \mathbf{U}^N(b)$. Suppose that $W(\mathbf{u}^N) > W(\mathbf{v}^N)$ for the RSWF $W : \mathbb{R}^N \to \mathbb{R}$ of Result 3. Because W is continuous, the intermediate value theorem implies the existence of a $\lambda \in (0, 1)$ such that $W(\hat{\mathbf{u}}^N) = \frac{1}{2}[W(\mathbf{u}^N) + W(\mathbf{v}^N)]$, where $\hat{\mathbf{u}}^N$ denotes the convex combination $(1-\lambda)\mathbf{u}^N + \lambda \mathbf{v}^N$. In particular $W(\mathbf{u}^N) > W(\hat{\mathbf{u}}^N) > W(\mathbf{v}^N)$. But the domain $\mathcal{D}^N$ is strongly saturating, so there must exist $c \in Z$ and $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ such that $\{\, a, c\,\}$ and $\{\, b, c\,\}$ are non-trivial pairs, while $\bar{\mathbf{U}}^N(a) = \mathbf{u}^N$, $\bar{\mathbf{U}}^N(b) = \mathbf{v}^N$, and $\bar{\mathbf{U}}^N(c) = \hat{\mathbf{u}}^N$. It follows that $a\ P(\bar{\mathbf{U}}^N)\ c$ and $c\ P(\bar{\mathbf{U}}^N)\ b$. But $P(\bar{\mathbf{U}}^N)$ is transitive, so $a\ P(\bar{\mathbf{U}}^N)\ b$. Then condition (I) implies that $a\ P(\mathbf{U}^N)\ b$. ∎

Now the three main theorems of Section 3 are obvious implications of Results 3 and 4.

5. Economic Environments

Suppose that the space of economic allocations involves finite sets G of private goods and H of public goods, at least one of which is non-empty. Here, a good is considered public if and only if several different individuals share a legitimate concern in the quantity that is made available by the economic, political, and social system. Two important special cases considered in the previous literature are **Case A** when $G = \emptyset$ and so there are only public goods, or **Case B** when $H = \emptyset$ and so there are only private goods. The general or mixed **Case C** occurs when $G \neq \emptyset$ and $H \neq \emptyset$, so there are both public and private goods. Although one can argue that this case has the most practical interest, so far only Bordes and Le Breton (1990a) have published results that concern it.

For each $i \in N$, let $X_i \subset \mathbb{R}^G$ denote the set of all possible allocations of private goods to individual i. Also, let $Y \subset \mathbb{R}^H$ denote the set of all possible allocations of public goods within the economy. Assume that each $i \in N$ has a feasible set $Z_i \subset X_i \times Y$ of possible combined allocations (x_i, y) of private and public goods.[9] Let $\mathbf{X}^N$ and $\mathbf{Z}^N$ denote the two Cartesian product sets $\prod_{i \in N} X_i$ and $\prod_{i \in N} Z_i$, with typical members $\mathbf{x}^N$ and $\mathbf{z}^N$ respectively. Then the underlying set of all possible social states in the form of individually feasible economic allocations is given by

$$Z := \{ (\mathbf{x}^N, y) \in \mathbf{X}^N \times Y \mid \forall i \in N : (x_i, y) \in Z_i \}$$

Additional resource balance and production constraints will limit the economy's choices to a feasible subset of Z. Social choice theory, however, is concerned with the specification of a social welfare ordering over the whole underlying set Z.

Each individual $i \in N$ is assumed to have some *privately oriented* utility function $U_i : Z_i \to \mathbb{R}$ defined on i's own feasible set $Z_i \subset X_i \times Y$; any such utility function is independent of the allocation to all other individuals of their own private goods. Then for each individual $i \in N$, the domain of all possible privately oriented utility functions is $\mathcal{U}(Z_i)$.

Assume that the restricted domain of each individual $i \in N$ satisfies $\mathcal{D}_i \subset \mathcal{U}(Z_i)$, and that the different individuals' utility domains $\mathcal{D}_i$ $(i \in N)$ are *independently restricted* in the

[9] It is standard in economic theory to regard such allocations as consumption vectors. The formulation here, however, allows each x_i, for instance, to be interpreted as a net trade vector in an economy with individual production.

sense that the *restricted domain* of allowable utility profiles is the Cartesian product set $\mathcal{D}^N := \prod_{i \in N} \mathcal{D}_i$.

In many economic contexts, each $\mathcal{D}_i$ might consist of continuous (or smooth) and/or (strictly) quasi-concave utility functions defined on Z_i. Each utility function may also satisfy suitable monotonicity properties. In particular, whenever $(x_i, y) \gg (x_i', y')$ in the strict vector ordering on $X_i \times Y \subset \mathbb{R}^G \times \mathbb{R}^H$, the pair $\{ (x_i, y), (x_i', y') \}$ will be trivial for i whenever preferences are restricted to be weakly monotone.

5.1. *Case A: Public Goods Only with a Common Utility Domain*

This first case was originally considered by Kalai, Muller, and Satterthwaite (1979). The underlying set Z is equal to Y, the domain of possible allocations of public goods.

In this case, our results use the following:

Domain Assumption (A). *All individuals $i \in N$ share a common saturating domain $\mathcal{D}_i = \mathcal{D}$, independent of i, of possible utility functions on the set $Y \subset \mathbb{R}^H$ of all possible public good vectors.*

That is, $\mathcal{D}$ would be a saturating domain if there were only one individual in the society. Obviously, because each individual $i \in N$ has the same domain $\mathcal{D}_i = \mathcal{D}$, the product domain $\mathcal{D}^N$ of utility profiles is also saturating. Because Result 3 applies only to non-trivial pairs, it is true as before without any need to assume a strongly saturating domain.

It remains to consider when Result 4 is true, since it applies to trivial pairs. Because $\mathcal{D}$ is a common saturating domain, if any pair $a, b \in Y$ is trivial, then all individuals must have exactly the same preference over $\{ a, b \}$ — that is, either $a \; P_i \; b$ for all $i \in N$, or $b \; P_i \; a$ for all $i \in N$, or $a \; I_i \; b$ for all $i \in N$. It follows that either $\mathbf{U}^N(a) \gg \mathbf{U}^N(b)$, or $\mathbf{U}^N(a) \ll \mathbf{U}^N(b)$, or $\mathbf{U}^N(a) = \mathbf{U}^N(b)$.

Without further assumptions such as the Pareto condition (P), an Arrow social welfare function displaying either universal indifference, or dictatorship, or inverse dictatorship on every non-trivial pair could still be arbitrary on one or more trivial pairs. Of course, by condition (I), it must give a constant preference on any such pair. For example, suppose that $Y = \mathbb{R}^H_+$, and that each utility function $U : Y \to \mathbb{R}$ in the common utility domain $\mathcal{D}$ is strictly monotone — i.e., $y > y'$ implies $U(y) > U(y')$ for all $y, y' \in \mathbb{R}^H_+$. Then $\{ 0, y \}$ is a trivial pair whenever $y \in \mathbb{R}^H_+$ with $y \neq 0$. In this case, there can be a dictator $d \in N$ who decides between all non-trivial pairs $y, y' \in \mathbb{R}^H_+ \setminus \{0\}$, while one can also have $0 \; P(\mathbf{U}^N) \; y$ for all $y \in \mathbb{R}^H_+ \setminus \{0\}$. This defines a non-dictatorial Arrow social welfare function satisfying (I) but violating (P). The same example shows that a welfarist ordering which is valid over non-trivial pairs need not extend to trivial pairs. But when condition (P) is imposed, more definite results can be obtained easily:

Result 4A. *Suppose that the SWFL F satisfies conditions (I) and (P) on a common saturating domain — i.e., domain assumption (A) is satisfied. Let $\{a, b\} \subset Z$ be any trivial pair. (a) If condition (CNC) is satisfied, then the common dictator of Result 3 is also a dictator for $\{a, b\}$. (b) If condition (P^0) is satisfied, then the welfarist ordering $\succsim$ of Result 3 also applies to $\{a, b\}$. (c) If condition (PC) is satisfied, then the RSWF $W : \mathbb{R}^N \to \mathbb{R}$ of Result 3 also applies to $\{a, b\}$.*

Proof: Because condition (P) is satisfied, on the trivial pair $\{a, b\}$ it must be true that either $\mathbf{U}^N(a) \gg \mathbf{U}^N(b)$ and $a \; P(\mathbf{U}^N) \; b$, or $\mathbf{U}^N(a) \ll \mathbf{U}^N(b)$ and $b \; P(\mathbf{U}^N) \; a$, or $\mathbf{U}^N(a) = \mathbf{U}^N(b)$. So (a) is true because each individual is a dictator on $\{a, b\}$. Also, (c) is true because, for a trivial pair $\{a, b\}$, it must be the case that $W(\mathbf{U}^N(a)) > W(\mathbf{U}^N(b))$ implies $\mathbf{U}^N(a) \gg \mathbf{U}^N(b)$ and so $a \; P(\mathbf{U}^N) \; b$.

As for (b), when condition (P^0) is also satisfied, on the trivial pair $\{a, b\}$ there are the following three possibilities: (i) $\mathbf{U}^N(a) \gg \mathbf{U}^N(b)$, so $\mathbf{U}^N(a) \succ \mathbf{U}^N(b)$ and $a \; P(\mathbf{U}^N) \; b$; or (ii) $\mathbf{U}^N(a) \ll \mathbf{U}^N(b)$, so $\mathbf{U}^N(b) \succ \mathbf{U}^N(a)$ and $b \; P(\mathbf{U}^N) \; a$; or (iii) $\mathbf{U}^N(a) = \mathbf{U}^N(b)$ and so, by (P^0), $a \; I(\mathbf{U}^N) \; b$. It follows that, even for trivial pairs $\{a, b\}$, one must have $\mathbf{U}^N(a) \succsim \mathbf{U}^N(b) \iff a \; R(\mathbf{U}^N) \; b$. ∎

Combined with Result 3, it follows that when domain assumption (A) is satisfied, Theorems 1, 2 and 3 are all valid provided that the Pareto condition (P) is also imposed as an extra assumption. No form of strong saturation is required here.

5.2. Case B: Private Goods Only

In this case the underlying set $Z = \mathbf{X}^N := \prod_{i \in N} X_i$, whose members are profiles of consumption vectors $\mathbf{x}^N = (x_i)_{i \in N}$ in the Cartesian product of different individuals' consumption sets X_i. Also, each individual i's domain $\mathcal{D}_i$ of allowable utility functions is some subset of the domain $\mathcal{U}(X_i)$ of all possible "selfish" utility functions $U_i(x_i)$ defined on X_i. Note that, because the sets $\mathcal{U}(X_i)$ are all different, so are the individuals' utility domains $\mathcal{D}_i$. This is unlike Case A which has only public goods and a common utility domain.

Even so, one can still consider an *individually saturating* domain of preferences $\mathcal{D}_i$ on X_i for every $i \in N$ separately — i.e., a domain such that X_i includes at least two individually non-trivial pairs relative to $\mathcal{D}_i$, and such that every individually non-trivial pair is connected to every other individually non-trivial pair by a chain of triples that are individually utility free, relative to $\mathcal{D}_i$. That is, $\mathcal{D}_i$ would be saturating if society consisted only of individual i.

With individually saturating domains, Results 1 and 2 of Section 4 are still true because they do not require any kind of saturating domain. Result 3, however, is generally false. To see why, first note that a pair $\{ \mathbf{a}^N, \mathbf{b}^N \} \subset \mathbf{X}^N$ is non-trivial in the domain $\mathcal{D}^N$ iff every pair $\{ a_i, b_i \} \subset X_i$ is individually non-trivial in the domain $\mathcal{D}_i$. Now, Result 3 relies on being able to connect any two non-trivial pairs $\{ \mathbf{a}^N, \mathbf{b}^N \}$ and $\{ \tilde{\mathbf{a}}^N, \tilde{\mathbf{b}}^N \}$ in $\mathbf{X}^N$ through a chain of utility free triples. Any such connection requires that, for each individual $i \in N$, the corresponding non-trivial pairs $\{ a_i, b_i \}$ and $\{ \tilde{a}_i, \tilde{b}_i \}$ in X_i be connected through a chain of individually utility free triples. Moreover, every individual's chain must be equally long. So far, none of our assumptions guarantee this; the chains of individually utility free triples which connect any two non-trivial pairs may have to be of different lengths for different individuals.

To circumvent this problem, Bordes and Le Breton (1989) strengthen the assumption that there are individually saturating domains. They require instead that there are (individually) *supersaturating* domains. A natural adaptation of this assumption to utility domains requires these to be individually saturating domains with the additional property that:

For each individually non-trivial pair $\{ a_i, b_i \} \subset X_i$*, there exists a disjoint non-trivial pair* $\{ \tilde{a}_i, \tilde{b}_i \} \subset X_i$ *such that all the four distinct three-member subsets* $\{ a_i, b_i, \tilde{a}_i \}$, $\{ a_i, b_i, \tilde{b}_i \}$, $\{ a_i, \tilde{a}_i, \tilde{b}_i \}$, $\{ b_i, \tilde{a}_i, \tilde{b}_i \}$ *of the set* $\{ a_i, b_i, \tilde{a}_i, \tilde{b}_i \}$ *are individually utility free triples.*

Obviously this is satisfied whenever $\{ a_i, b_i, \tilde{a}_i, \tilde{b}_i \}$ is a utility free quadruple, but our assumption is somewhat weaker.

The role of the supersaturation assumption is to allow different individuals' chains of connecting individually utility free triples to be extended until they all become equally long. The assumption works by allowing any individual's non-trivial pair $\{ a_i, b_i \}$ to be connected to itself through either of the following two looped chains of free triples:

$$\begin{aligned} &\{ a_i, b_i, \tilde{a}_i \}, \{ b_i, \tilde{a}_i, a_i \}, \{ \tilde{a}_i, a_i, b_i \}; \\ \text{or} \quad &\{ a_i, b_i, \tilde{a}_i \}, \{ b_i, \tilde{a}_i, \tilde{b}_i \}, \{ \tilde{a}_i, \tilde{b}_i, a_i \}, \{ \tilde{b}_i, a_i, b_i \}. \end{aligned}$$

By adding enough of these looped chains of length three or four separately to each individual's connecting chain, all these chains can be made exactly the same length. For this reason, if each individual's utility domain $\mathcal{D}_i$ is individually supersaturating, the Cartesian product domain $\mathcal{D}^N$ of utility function profiles is saturating, according to the definition of Section 2. Therefore, Result 3 of Section 4 applies here also. Thus, Result 3 becomes true in individually supersaturating domains of selfish preferences for private goods.

In order to accommodate trivial pairs and so make Result 4 true in this case as well, Bordes and Le Breton (1989) introduce another strengthening of individual saturation, resulting in an (individually) *hypersaturating domain*. For the case of utility domains, hypersaturation requires that each $\mathcal{D}_i$ be supersaturating, and also that, whenever $\{ a_i, b_i \} \subset X_i$ is an individually trivial pair and $U_i \in \mathcal{D}_i$, there exists $c_i \in X_i$ such that $\{ a_i, c_i \}$ and $\{ b_i, c_i \}$ are non-trivial pairs, while for any $\lambda \in [0, 1]$, there exists a utility function $\bar{U}_i \in \mathcal{D}_i$ satisfying:

$$\bar{U}_i(a_i) = U_i(a_i); \quad \bar{U}_i(b_i) = U_i(b_i); \quad \bar{U}_i(c_i) = (1 - \lambda)U_i(a_i) + \lambda U_i(b_i).$$

Obviously, if each individual domain $\mathcal{D}_i$ is hypersaturating, then the Cartesian product domain $\mathcal{D}^N$ is strongly saturating.

Domain Assumption (B). *Each individual* $i \in N$ *has a hypersaturating domain* $\mathcal{D}_i$ *of possible utility functions on the set* $X_i \subset \mathbb{R}^G$ *of all possible private good vectors.*

Under this assumption, clearly Theorems 1–3 all hold for a private good economy of the kind described above.

In fact, only Theorem 3 requires the full strength of hypersaturation. As in Section 3, Theorems 1 and 2 are still true under weaker versions of hypersaturation.

Specifically, the individual domain $\mathcal{D}_i$ is *ordinally hypersaturating* if it is supersaturating and also, whenever $\{ a_i, b_i \} \subset X_i$ is an individually trivial pair and $U_i \in \mathcal{D}_i$ satisfies $U_i(a_i) > U_i(b_i)$, then there exist $c_i \in X_i$ and a utility function $\bar{U}_i \in \mathcal{D}_i$ such that $\{ a_i, c_i \}$ and $\{ b_i, c_i \}$ are non-trivial pairs, while $\bar{U}_i(a_i) = U_i(a_i) > \bar{U}_i(c_i) > U_i(b_i) = \bar{U}_i(b_i)$.

On the other hand, the individual domain $\mathcal{D}_i$ is *less hypersaturating* if it is supersaturating and also, whenever $\{ a_i, b_i \} \subset X_i$ is an individually trivial pair and $U_i \in \mathcal{D}_i$, then there exist $c_i \in X_i$ and a utility function $\bar{U}_i \in \mathcal{D}_i$ such that $\{ a_i, c_i \}$ and $\{ b_i, c_i \}$ are non-trivial pairs, while $\bar{U}_i(a_i) = \bar{U}_i(c_i) = U_i(a_i)$ and $\bar{U}_i(b_i) = U_i(b_i)$.

Clearly, these last two definitions imply that if each individual domain $\mathcal{D}_i$ is ordinally (resp. less) hypersaturating, then the Cartesian product domain $\mathcal{D}^N$ is ordinally (resp. less) strongly saturating, so Theorem 1 (resp.Theorem 2) must hold.

As an example, suppose that $X_i = \mathbb{R}^G_+$ where $\#G \geq 2$, and that all utility functions in $\mathcal{D}_i$ are strictly increasing. If $\mathcal{D}_i$ is sufficiently rich, it will obviously be supersaturating — for example, it is enough for $\mathcal{D}_i$ to include all monotone strictly quasi-concave utility functions. Note, however, that $\{ 0, x_i \}$ is a trivial pair whenever $x_i > 0$. It is easy to check that this prevents each $\mathcal{D}_i$ from being hypersaturating. Indeed, it cannot be ordinally or less hypersaturating either. In order to exclude a famous example due to Blau (1957) — see also Border (1983) — one should consider $X_i = \mathbb{R}^G_+ \setminus \{0\}$ instead, on which the set of monotone utility functions is a hypersaturating domain, as is the set of monotone (strictly) quasi-concave utility functions. Accordingly, Theorems 1–3 all hold for this hypersaturating domain.[10]

[10] See Campbell (1989a, b) for an exploration of the possible social welfare functions without interpersonal comparisons when 0 is retained in X_i.

5.3. Case C: Mixed Public and Private Goods

This is the most interesting case, in which the underlying set is $Z = \mathbf{X}^N \times Y := \prod_{i \in N} X_i \times Y$, whose members $(\mathbf{x}^N, y)$ are profiles of consumption vectors $\mathbf{x}^N = (x_i)_{i \in N}$ in the Cartesian product of different individuals' consumption sets X_i, combined with a public good vector $y \in Y$. Also, each individual i's restricted utility domain $\mathcal{D}_i$ is a subset of the respective domain $\mathcal{U}(X_i \times Y)$ that consists of individual utility functions $U_i(x_i, y)$ defined on $X_i \times Y$.

In Case B we provided sufficient conditions to ensure that the product domain is strongly saturating, in the appropriate sense. In Case C, however, our sufficient conditions will not ensure this, as a later example will show.

Indeed, our domain assumptions seem natural adaptations of those used by Bordes and Le Breton (1990a) to define an "ultrasaturating" domain in which they prove Arrow's and related theorems. The assumptions are stated more easily if one begins by defining, for each $y \in Y$ and each individual $i \in N$ with utility domain $\mathcal{D}_i$, the *conditional* (on y) *relative to the private component* (or RPC) utility domain as

$$\mathcal{D}_i^y := \{ U_i^y : X_i \to \mathbb{R} \mid \exists U_i \in \mathcal{D}_i : U_i^y(\cdot) \equiv U_i(\cdot, y) \} \subset \mathcal{U}(X_i)$$

Then, let $\mathcal{D}_i^* := \cup_{y \in Y} \mathcal{D}_i^y \subset \mathcal{U}(X_i)$ be i's *RPC utility domain.* It consists of all utility functions for private goods which, for some $y \in Y$, are compatible with the domain $\mathcal{D}_i$. Let $\mathcal{D}^{*N} := \prod_{i \in N} \mathcal{D}_i^*$ denote the Cartesian product of all the RPC domains $\mathcal{D}_i^*$.

Next, say that the set $S \subset X_i \times Y$ is an *RPC non-trivial pair* (resp. an *RPC utility free triple*) for individual i if and only if the projection

$$S_i = \{ x_i \in X_i \mid \exists y \in Y : (x_i, y) \in S \}$$

of S onto X_i is a non-trivial pair (resp. a utility free triple) relative to the domain $\mathcal{D}_i^*$. Thus, S is an RPC utility free triple for i iff $\cup_{y \in Y} \mathcal{D}_{i \, | S_i}^y = \mathcal{U}(S_i)$, and S is an RPC non-trivial pair for i iff $\# \cup_{y \in Y} \psi(\mathcal{D}_{i \, | S_i}^y) \geq 2$. For this specific domain, our results use the following assumptions:

Domain Assumptions (C).

(C.1) For each $i \in N$, the RPC utility domain $\mathcal{D}_i^*$ is supersaturating on X_i.

(C.2) *For each* $i \in N$, *if* $S \subset X_i \times Y$ *is an RPC non-trivial pair (resp. an RPC utility free triple), then* S *is also non-trivial (resp. utility free) relative to* $\mathcal{D}_i$.

(C.3) *Given any* $y, y' \in Y$, *there exists* $\bar{y} \in Y$ *such that for all* $i \in N$, *whenever either* $a_i = b_i$ *or the pair* $\{ a_i, b_i \}$ *in* X_i *is trivial relative to* $\mathcal{D}_i^*$, *one can find* $c_i \in X_i$ *for which* $\{ (a_i, y), (c_i, \bar{y}) \}$ *and* $\{ (c_i, \bar{y}), (b_i, y') \}$ *are non-trivial pairs relative to* $\mathcal{D}_i$ *and also, given any* $U_i \in \mathcal{D}_i$ *and any* $\lambda \in [0, 1]$, *there exists a utility function* $\bar{U}_i \in \mathcal{D}_i$ *satisfying:*

$$\bar{U}_i(a_i, y) = U_i(a_i, y);\ \bar{U}_i(b_i, y') = U_i(b_i, y');\ \bar{U}_i(c_i, \bar{y}) = (1 - \lambda)U_i(a_i, y) + \lambda U_i(b_i, y') \quad (1)$$

Here, assumption (C.1) is an obvious extension to utility domains of a combination of conditions (4), (5) and (7) in Bordes and Le Breton (1990a, p. 8). Similarly, assumption (C.2) amounts to an obvious extension of a combination of their conditions (1) and (2). It requires that, in order for the set S to be non-trivial (resp. utility free), it is sufficient (though not necessary) that S be non-trivial (resp. utility free) when only variations in private goods are considered. That leaves (C.3), which replaces their conditions (3) and (6) with one new assumption having much of the flavour of the separation condition that was introduced in Section 2. Condition (C.3) also plays a similar role to the extension from supersaturating to hypersaturating domains that was considered in Case B; however, because of its treatment of the public component, requiring in particular the existence of a $\bar{y} \in Y$ which is independent of i, condition (C.3) is obviously stronger than merely requiring each RPC utility domain $\mathcal{D}_i^*$ to be hypersaturating on X_i.

Obvious weakenings of condition (C.3) would still allow Wilson's Theorem or strong welfarism to be proved, just as ordinal or less strong saturation was enough in Section 4.

Note that, whenever $y \in Y$ and $S_i \subset X_i$ is a non-trivial pair (resp. a utility free triple) relative to $\mathcal{D}_i^*$, the set $S_i \times \{y\}$ is an RPC non-trivial pair (resp. an RPC utility free triple). Because of (C.2), it follows that $S_i \times \{y\}$ is also a non-trivial pair (resp. a utility free triple) relative to $\mathcal{D}_i$, and so S_i is a non-trivial pair (resp. a utility free triple) relative to $\mathcal{D}_i^y$. Evidently, therefore, (C.2) implies that (C.1) is satisfied iff $\mathcal{D}_i^y$ is supersaturating for all $i \in N$ and all $y \in Y$. Adding condition (C.3) ensures that each such $\mathcal{D}_i^y$ is hypersaturating when the public component y remains fixed, but it also allows for variations in y in a way which makes the main theorems true. Of course, when $Y = \{0\}$ so that in effect there are only private goods, then Domain Assumptions (C) together give one a hypersaturating domain, exactly as in Case B.

An easier argument than that set out in Bordes and Le Breton (1990a, Section 6) establishes that all three assumptions (C.1)–(C.3) are satisfied in the "standard" case where, for all $i \in N$, the domain $\mathcal{D}_i$ consists of all continuous, strictly increasing and quasi-concave utility functions on the familiar product space $X_i \times Y$ with $X_i = \mathbb{R}^G_+ \setminus \{0\}$ (where $\#G \geq 2$) and $Y = \mathbb{R}^H_+$. As explained in our discussion of Case B, it is important to exclude 0 from $\mathbb{R}^G_+$.

The following is the promised example showing that the domain assumptions (C) can be satisfied by a non-saturating domain.

Example.

Suppose that $N = \{1, 2\}$, so there are 2 individuals. Suppose too that $X = X_1 = X_2 = \{a, b, c\}$ and $Y = \{y', y''\}$. Suppose that an SWFL satisfying (ONC) is defined on the utility domain $\mathcal{D}_1 \times \mathcal{D}_2$, where each $\mathcal{D}_i$ consists of all utility functions U_i representing preference orderings R_i on $X_i \times Y$ such that either $(a, y')\ R_i\ (a, y'')$ or else, if $(a, y'')\ P_i\ (a, y')$, then $(x, y)\ R_i\ (a, y')$ for all $(x, y) \in X_i \times Y$.

First, note that $\mathcal{D}^y_i = \mathcal{D}^*_i = \mathcal{U}(X_i)$ for all $i \in N$ and all $y \in Y$. So the RPC utility domain $\mathcal{D}^*_i$ is unrestricted on X_i, not just supersaturating. This verifies condition (C.1).

Second, every pair in $X_i \times Y$ is evidently both utility free and RPC utility free. In particular, there are no trivial pairs or RPC trivial pairs. Also, for each $y_1, y_2, \bar{y} \in Y$ and $x \in X$, there exists $\bar{x} \in X \setminus \{x, a\}$ such that, given any $U_i \in \mathcal{D}_i$ and any $\lambda \in [0, 1]$, there exists a utility function $\bar{U}_i \in \mathcal{D}_i$ satisfying

$$\bar{U}_i(x, y_1) = U_i(x, y_1);\ \bar{U}_i(x, y_2) = U_i(x, y_2);\ \bar{U}_i(\bar{x}, \bar{y}) = (1 - \lambda)U_i(x, y_1) + \lambda U_i(x, y_2)$$

Note too that this is possible even when $\{(x, y_1), (x, y_2), (\bar{x}, \bar{y})\}$ is not a free triple because $x = a$ or $y_1 = y_2$. This verifies condition (C.3).

Third, note that any triple $T \subset X_i \times Y$ is utility free if and only it does not include the pair $\{(a, y'), (a, y'')\}$ as a subset. This implies that neither domain $\mathcal{D}_i$ is saturating because the free pair $\{(a, y'), (a, y'')\}$ is not part of any utility free triple, and so cannot be linked to any of the other free pairs by a chain of utility free triples. On the other hand, any triple $S \subset X_i \times Y$ is RPC utility free if and only if its projection onto X_i is the triple $\{a, b, c\}$. But then S must be utility free. Together with the absence of trivial pairs, this verifies condition (C.2).

The three domain assumptions (C), though weaker than saturation, still allow us to prove new versions of Results 3 and 4, with non-trivial pairs replaced by RPC non-trivial pairs, and with trivial pairs replaced by RPC trivial pairs. Specifically:

RESULT 3C. *Suppose that both domain assumptions (C.1) and (C.2) are satisfied. Then Result 3 is true for all RPC non-trivial pairs* $\{ (\mathbf{a}^N, y), (\mathbf{b}^N, y') \}$ *in* $Z = \mathbf{X}^N \times Y$.

PROOF: First, (C.1) implies that for all $i \in N$, the space X_i includes at least two disjoint pairs $\{ a_i, b_i \}, \{ c_i, d_i \}$ that are non-trivial relative to $\mathcal{D}_i^*$. Then, for any $y \in Y$, each pair $\{ (a_i, y), (b_i, y) \}, \{ (c_i, y), (d_i, y) \} \subset X_i \times Y$ is RPC non-trivial and so, by (C.2), non-trivial relative to $\mathcal{D}_i$. It follows that the pairs $\{ (\mathbf{a}^N, y), (\mathbf{b}^N, y) \}, \{ (\mathbf{c}^N, y), (\mathbf{d}^N, y) \} \subset Z$ are different and also non-trivial relative to $\mathcal{D}^N$.

Second, suppose that the two pairs $\{ (\mathbf{a}^N, y_1), (\mathbf{b}^N, y_2) \}, \{ (\mathbf{c}^N, y_3), (\mathbf{d}^N, y_4) \} \subset Z$ are both RPC non-trivial. By definition, the pairs $\{ \mathbf{a}^N, \mathbf{b}^N \}, \{ \mathbf{c}^N, \mathbf{d}^N \} \subset \mathbf{X}^N$ must be non-trivial relative to $\mathcal{D}^{*N}$. By (C.1) and the argument given for Case B when there are only private goods, it follows as in the proof of Result 3 in Section 4 that $\{ \mathbf{a}^N, \mathbf{b}^N \}$ can be connected to $\{ \mathbf{c}^N, \mathbf{d}^N \}$ by a finite chain of overlapping triples $T_k \subset \mathbf{X}^N$ $(k = 1, 2 \ldots, r)$ which are utility free relative to $\mathcal{D}^{*N}$, with $\{ \mathbf{a}^N, \mathbf{b}^N \} \subset T_1$ and $\{ \mathbf{c}^N, \mathbf{d}^N \} \subset T_r$. Then there exists a corresponding finite chain of overlapping triples $V_k \subset Z$ $(k = 1, 2 \ldots, r)$ with $\{ (\mathbf{a}^N, y_1), (\mathbf{b}^N, y_2) \} \subset V_1$ and $\{ (\mathbf{c}^N, y_3), (\mathbf{d}^N, y_4) \} \subset V_r$, such that the projection of each V_k onto $\mathbf{X}^N$ is T_k. Because each T_k is utility free relative to $\mathcal{D}^{*N}$, it follows by definition that each V_k is RPC utility free. Hence, by (C.2), each V_k is utility free relative to $\mathcal{D}^N$. In this way, all RPC non-trivial pairs in Z can be connected by a finite chain of triples which are utility free relative to $\mathcal{D}^N$.

The rest of the proof is the same as that of parts (a), (b) and (c) of Result 3 in Section 4. ▮

RESULT 4C. *Suppose that all three domain assumptions (C.1)–(C.3) are satisfied. Then Result 4 is true for all RPC trivial pairs* $\{ (\mathbf{a}^N, y), (\mathbf{b}^N, y') \}$ *in* $Z = \mathbf{X}^N \times Y$.

PROOF: Suppose that the pair $\{ (\mathbf{a}^N, y), (\mathbf{b}^N, y') \}$ is RPC trivial because either $a_i = b_i$ or $\{ a_i, b_i \} \subset X_i$ is trivial relative to $\mathcal{D}_i^*$ for $i \in N' \neq \emptyset$, whereas $\{ a_i, b_i \}$ is non-trivial relative to $\mathcal{D}_i^*$ for $i \in N \setminus N'$ (which may be empty). By (C.3), there exists $\bar{y} \in Y$ and also, for each $i \in N'$, some $c_i \in X_i$ such that $\{ (a_i, y), (c_i, \bar{y}) \}$ and $\{ (c_i, \bar{y}), (b_i, y') \}$ are non-trivial relative to $\mathcal{D}_i$, while for any $U_i \in \mathcal{D}_i$ and any $\lambda \in [0, 1]$, there exists a utility function $\bar{U}_i \in \mathcal{D}_i$ satisfying equations (1).

Also, by (C.1), for each $i \in N \setminus N'$, there exists $c_i \in X_i$ such that $\{ a_i, b_i, c_i \}$ is utility free relative to $\mathcal{D}_i^*$. Then the triple $\{ (a_i, y), (b_i, y'), (c_i, \bar{y}) \}$ is RPC utility free, so (C.2) implies that it is utility free relative to $\mathcal{D}_i$. Hence, for any $U_i \in \mathcal{D}_i$ and any $\lambda \in [0,1]$, there exists a utility function $\bar{U}_i \in \mathcal{D}_i$ satisfying equations (1) in this case also. Thus, for any utility profile $\mathbf{U}^N \in \mathcal{D}^N$ and any $\lambda \in [0,1]$, there exists $\bar{\mathbf{U}}^N \in \mathcal{D}^N$ such that:

$$\bar{\mathbf{U}}^N(\mathbf{a}^N, y) = \mathbf{U}^N(\mathbf{a}^N, y); \quad \bar{\mathbf{U}}^N(\mathbf{b}^N, y') = \mathbf{U}^N(\mathbf{b}^N, y');$$
$$\bar{\mathbf{U}}^N(\mathbf{c}^N, \bar{y}) = (1-\lambda)\mathbf{U}^N(\mathbf{a}^N, y) + \lambda \mathbf{U}^N(\mathbf{b}^N, y').$$

The rest of the proof is the same as that of parts (a), (b) and (c) of Result 4 in Section 4. ∎

It follows finally that the three domain assumptions (C.1)–(C.3) together can replace condition (U) in the theorems due to Wilson, to d'Aspremont and Gevers, and to Roberts, as well in Arrow's impossibility theorem — including the version with cardinal non-comparable utility functions due to Sen (1970).

Appendix

Lemma. (cf. Sen, 1970, Theorem 8*2, pp. 129–30.) *For any utility free set S with $\#S \geq 2$, if the SWFL $F : \mathcal{D}^N \to \mathcal{R}(S)$ satisfies conditions (I) and (CNC) on S, then it also satisfies (ONC) on S.*

Proof: Consider any pair of utility function profiles $\mathbf{U}^N, \bar{\mathbf{U}}^N \in \mathcal{D}^N$ which are ordinally equivalent on S because the restrictions to S of the corresponding orderings are equal — i.e., $\psi^N(\mathbf{U}^N)_{|S} = \psi^N(\bar{\mathbf{U}}^N)_{|S}$. Let $a, b \in S$ be any pair, and suppose that $a\ R(\mathbf{U}^N)\ b$. Now, for each $i \in N$, define the two constants

$$\rho_i := \begin{cases} \dfrac{\bar{U}_i(a) - \bar{U}_i(b)}{U_i(a) - U_i(b)} & \text{if } U_i(a) \neq U_i(b) \\ 1 & \text{if } U_i(a) = U_i(b) \end{cases} \qquad \text{and} \quad \alpha_i := \bar{U}_i(b) - \rho_i\, U_i(b)$$

Because $\mathbf{U}^N$ and $\bar{\mathbf{U}}^N$ are ordinally equivalent on S, it follows that $\rho_i > 0$ and also that $\bar{U}_i(x) = \alpha_i + \rho_i\, U_i(x)$ for $x \in \{a, b\}$. Because S is utility free, there exists a utility profile $\hat{\mathbf{U}}^N \in \mathcal{D}^N$ such that $\hat{U}_i(x) := \alpha_i + \rho_i\, U_i(x)$ for all $x \in S$. Now $\mathbf{U}^N$ and $\hat{\mathbf{U}}^N$ are cardinally equivalent on S, so condition (CNC) on S implies that $F(\mathbf{U}^N)_{|S} = F(\hat{\mathbf{U}}^N)_{|S}$. In particular, $a\ R(\hat{\mathbf{U}}^N)\ b$. Finally, $\bar{\mathbf{U}}^N|_{\{a,b\}} = \hat{\mathbf{U}}^N|_{\{a,b\}}$, so condition (I) on S implies that $a\ R(\bar{\mathbf{U}}^N)\ b$.

Conversely, if $a\ R(\bar{\mathbf{U}}^N)\ b$, interchanging $\mathbf{U}^N$ and $\bar{\mathbf{U}}^N$ in this argument shows that $a\ R(\mathbf{U}^N)\ b$. Therefore $a\ R(\mathbf{U}^N)\ b \iff a\ R(\bar{\mathbf{U}}^N)\ b$. Because this is true for all pairs $a, b \in S$, it follows that F satisfies (ONC) on S. ∎

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