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The power of small coalitions in large economies

Peter J. Hammond
Stanford University

Chapter 3. *Mathematical Models in Economics*. MOL Bacharach, MAH Dempster, and JL Enos, eds. University of Oxford (1990).

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Abstract

Economies with a continuum of agents were introduced to mathematical economics by Aumann (1964, 1966). They make precise the ideal that the core of an economy will coincide with the set of Walrasian allocations in an economy with a 'large' number of traders. A few years later related work, especially the early papers of Hildenbrand on this topic, were discussed in the mathematical economics seminar in Oxford, often before publication. Now Hildenbrand (1974, 1982) and Mas-Colell (1985) present most of the relevant results and literature except the most recent. Here I discuss some later work on finite coalitions in continuum economies, and try to relate it to the existing literature.

1. Introduction and preliminaries

1.A Continuum economies and negligible sets of agents

A continuum economy involves a non-atomic measure space of agents $(A, \boldsymbol{A}, \mu)$. Little is lost by taking A as the unit interval $[0,1]$ in $\mathbb{R}$, $\boldsymbol{A}$ as the σ-algebra of Lebesgue measurable sets, and μ as Lebesgue measure. The economy is described by a measurable mapping $\boldsymbol{E}: A \to \Theta$

This work was supported by the National Science Foundation. The paper was presented to the Mathematical Economics seminar at Stanford University on 3 April 1986 and the Roy–Malinvaud seminar in Paris, 12 May 1986. Parts are based on earlier work I have conducted jointly with Mamoru Kaneko and Myrna Wooders. I am grateful to both for many enlightening discussions including comments on the first version of this paper. We all three owe much to Robert Aumann, Paul Champsaur, Birgit Grodal, Mordecai Kurz and Guy Laroque for comments on this and earlier work to which this is intended, in part, as a reply. And I am also most grateful to John Hillas for a suggestion which led to the result of §3.D, to Nick Baigent for drawing my attention to the special relevance of Kannai's work, to Ed Green, Joseph Greenberg, Dieter Sondermann, and Yair Tauman. None of these bears responsibility for errors in what follows.

from the set of agents to the set Θ of agents' characteristics—endowments and consumption sets, typically—equipped with a σ-algebra that is usually the Borel σ-algebra, or the completion of that algebra, when Θ has been given a topology (see especially Hildenbrand 1974). An allocation in the economy is described by a measurable function $f : A \to X$ where X is the commodity vector space—usually a subset of $\mathbb{R}^\ell$, where ℓ is the finite number of different commodities.

In order, apparently, to apply the mathematical theory of $\mathcal{L}_p$ function spaces, which are metric spaces, it has become standard to regard the allocation f as defined only up to a set of measure zero. Specifically, two allocations f_1, f_2, are treated as equivalent whenever $\alpha(\{a \in A | f_1(a) \neq f_2(a)\}) = 0$—i.e. $f_1(a) = f_2(a)$ a.e. ('almost everywhere') in A. Equivalently, the allocations f_1 and f_2 are equivalent if and only if, for each measurable coalition C, a subset of A:

$$\int_C f_1 \, d\mu = \int_C f_2 \, d\mu$$

so that an allocation corresponds uniquely to an absolutely continuous vector measure defined on the σ-algebra of all measurable coalitions.

All such absolutely continuous measures allocate zero to a coalition of measure zero—in particular, to each individual and each coalition of a finite number of agents. Thus the identity of each agent is submerged completely in the non-atomic measure. This has led Aumann, for one, to regard an agent not as a point a of the set A, but as an interval da of arbitrary small measure. And to allow as blocking coalitions only sets with positive measure. Thus, only a subcontinuum of agents has any power to alter the allocation.

1.B The power of finite coalitions in continuum economies

Ignoring coalitions of measure zero undervalues the economic power of individuals and small groups. Most trade is bilateral, with goods exchanged for money. The owners of a typical business are a finite coalition of measure zero. Traders acting individually can bring about Pareto efficient allocations through Walrasian markets.

For example, consider a two-class economy, with α-agents and β-agents in equal numbers. Suppose the exchange economy with one α-agent and one β-agent has a unique Walrasian equilibrium. Then their respective equilibrium net trade vectors x^α, x^β satisfy $x^\alpha + x^\beta = 0$. In the replica economy with r agents of each type, a replica of the same Walrasian equilibrium can be achieved by having α-agents and β-agents pair off to trade. The r coalitions of size two still have the same power to achieve this equilibrium, no matter how large r

may be. In the limit, with a continuum of α-agents and β-agents of equal measure, traders can still pair off in a continuum of coalitions to achieve an equivalent Walrasian equilibrium.

Indeed, a continuum economy is a mathematical abstraction, representing the limit of a sequence of economies in which the number of agents has become infinitely large. The measure of a set of agents represents a proportion of the (large) total number of agents. Now, as the economy expands, the power of each individual, or of each finite coalition, does become less. But if one doubles the *number* of finite coalitions who exercise power as the economy doubles in size, the total effect does not approach zero; there is no need to double the *size* of each coalition. In the limit, one has a continuum of finite coalitions which together have power, rather than one continuum coalition.

This idea should not be unfamiliar. For the standard definition of a Walrasian equilibrium allocation in a continuum economy recognizes the power of individuals. Provided that, at a given price vector, a set of individuals of positive measure can upset an allocation by finding superior consumption vectors in their budget set, that ensures that the allocation is not a Walrasian equilibrium at that price vector.

Our joint paper Hammond, Kaneko and Wooders (1989, hereafter referred to simply as HKW) considers a new concept of the core for a continuum economy. This is the 'f-core', in which only finite coalitions are allowed.[1] Though the definition is quite different from the usual core, we proved that there is the same equivalence between the f-core and the set of Walrasian allocations as there is for the usual core.

To lend these results more significance, corresponding limit theorems should be demonstrated for large finite economies. This is done here and in a companion paper by Kaneko and Wooders (1989). For the most part, I consider economies in which agents have preferences only for their own consumption, whereas Kaneko and Wooders follow HKW in considering 'widespread externalities' as well.

1.C Finite coalitions in large finite economies

In a continuum economy, an 'Aumann' coalition C is a measurable set of agents which is not null—i.e. $\mu(C) > 0$. Let $\boldsymbol{E}_n$ $(n = 1, 2, \ldots)$ be a sequence of large finite economies in which the number of agents $\#\boldsymbol{E}_n$ converges to infinity. If this sequence converges to the continuum economy $\boldsymbol{E}$, the 'Aumann' coalition C is the limit of a sequence C_n

[1] In choosing this terminology, we unfortunately overlooked Kannai's (1970, p. 793) different prior use, which seems, however, not to have been adopted since, so we hope that no confusion results.

$(n = 1, 2, \ldots)$ of coalitions in each finite economy of the sequence of economies. In this sequence, the proportion of agents $\#C_n/\#\boldsymbol{E}_n$ in each economy $\boldsymbol{E}_n$ who belong to the corresponding coalition C_n converges to $\mu(C) > 0$. In particular, the proportion is bounded away from zero as the economy becomes large. By contrast, a finite coalition F of the continuum economy, with m members, say, is the limit of a sequence of finite coalitions F_n $(n = 1, 2, \ldots)$ in each finite economy $\boldsymbol{E}_n$ of the convergent sequence of economies, with each coalition F_n of the sequence having no more than m members. So the proportion of agents in each coalition of the sequence converges to zero as the economy becomes large. There lies the crucial difference between the f-core and the usual core.

The f-core of a continuum economy allows coalitions of arbitrarily large finite size. So, if $\boldsymbol{E}_n$ $(n = 1, 2, \ldots)$ is a sequence of finite economies converging to the continuum economy $\boldsymbol{E}$, the maximum size m_n of an allowable coalition in $\boldsymbol{E}_n$ must be unbounded if a limit theorem is to hold. Indeed, the limit theorems stated below concern 'm_n-cores' of finite economies, in which only coalitions of size not exceeding m_n are allowed to form $(n = 1, 2, \ldots)$, and where $m_n \to \infty$ as $n \to \infty$. Thus allowable coalitions become arbitrarily large, even though $m_n/\#\boldsymbol{E}_n$ may tend to zero as $n \to \infty$ and may even tend to zero very fast.

1.D Outline

Section 2 contains some preliminary notation, definitions and assumptions. Thereafter, §3 considers limit theorems for the core of replica exchange economies. The 'first m-core' is defined as the set of allocations that cannot be blocked except perhaps by coalitions whose size exceeds m agents. The 'm-core' is defined as the set of allocations in the first m-core which can be achieved by exchange within coalitions whose size does not exceed m agents. The limit theorem considers the m_n-cores of a sequence $\boldsymbol{E}_n$ $(n = 1, 2, \ldots)$ of economies where both m_n and $\#\boldsymbol{E}_n$ tend to infinity as n tends to infinity. The theorem shows that, under the usual assumptions for standard limit theorems, the m_n-core of $\boldsymbol{E}_n$ converges to the core of the limit economy as n tends to infinity. This is true even though $m_n/\#\boldsymbol{E}_n$ may converge to zero. Thus coalitions with an arbitarily small proportion of all agents are powerful enough to shrink the core in the usual way as the economy becomes large.

Section 4 uses a result of Mas-Colell (1979) to prove a limit theorem like one of Anderson (1978) when, as in §2, coalitions are bounded but the bound tends to infinity as the economy becomes infinitely large. It seems that all the limit results of Hildenbrand (1974, 1982) and Mas-Colell (1985) will apply similarly. The section concludes with another limit theorem similar to that of Kannai (1970, Theorem A).

Section 5 turns to continuum economies and the equivalence of the core and the set of Walrasian allocations. It is shown that the usual equivalence theorem is still true even if a blocking coalition has to be partitioned in a 'measure-consistent' manner into a continuum of finite sub-coalitions, and if trade can only take place among the members of each such finite sub-coalition. The only qualification concerns the difference between the 'first core' of unblocked allocations and the 'core' of allocations achievable within finite coalitions which are unblocked. This result concerning the 'f^*-core' of a continuum economy links the work of HKW to the usual Aumann equivalence theorem. Indeed it is a stepping stone toward the main result of HKW (in the absence of widespread externalities), as reviewed in §6, concerning the equivalence of the set of Walrasian allocations and the 'f-core.'

Until §6, all the results amount to a slight strengthening of standard results, allowing restrictions on the maximum size of a coalition. Section 6 also reviews the application of our new notion of the f-core to an economy with 'widespread externalities,' which yields an equivalence theorem where none is available with the usual concept of the core. In similar vein, attention is also drawn to another equivalence result concerning multilateral incentive compatibility in continuum economies (Hammond 1987).

Section 7 contains concluding remarks.

2. Preliminary notation, definitions, and assumptions

X will denote the *commodity space*, which will be a subset of $\mathbb{R}^\ell$ for some finite number of commodities ℓ. There will be a set of *agents* A which will either be finite, or $(A, \boldsymbol{A}, \mu)$ will be a non-atomic measure space, in which A will be $[0,1]$ and μ will be Lebesgue measure on the real line $\mathbb{R}$, with $\boldsymbol{A}$ the Lebesgue σ-algebra, both restricted to $[0,1]$.

Each agent $a \in A$ is assumed to have a *consumption set* $X(a)$ which is a subset of X, and closed, convex and bounded below. Each agent $a \in A$ is also assumed to have an *endowment* $e(a) \in X$ and a continuous transitive complete *preference* ordering $\succsim_a$ which can be regarded as a closed subset of $X(a) \times X(a)$. An *economy* $\boldsymbol{E}$ will be a mapping $\boldsymbol{E} : A \to \Theta$ where Θ denotes the space of *characteristics* of consumers—namely combinations $(X(a), \succsim_a, e(a))$ consisting of a consumption set, preference relation and endowment vector. When

$(A, \boldsymbol{A}, \alpha)$ is a non-atomic measure space, $\boldsymbol{E}$ is a *continuum economy* if Θ is given the topology of closed convergence (Hildenbrand 1974) and $\boldsymbol{E}^{-1}(K)$ is $\boldsymbol{A}$-measurable whenever K is Borel measurable in Θ.

An *allocation* is a mapping $f : A \to X$ which, in the case of a continuum economy, must be measurable, and in all cases must satisfy:

(i) $f(a) \in X(a)$ (a.e. in A)

(ii) $\sum_{a \in A} [f(a) - e(a)] \leq 0$ (in a finite economy)

or $\int_A (f - e) \leq 0$ (in a continuum economy).

An allocation f is *Walrasian* in the economy $\boldsymbol{E}$ if there is a price vector $p > 0$ such that, a.e. in A:

(i) $pf(a) \leq pe(a)$

(ii) $x \succ_a f(a)$ implies $px > pe(a)$

(iii) $p \int_A (f - e) = 0$.

Write $W(\boldsymbol{E})$ for the set of Walrasian allocations in the economy $\boldsymbol{E}$.

In some sections, further assumptions will be needed. Preferences are said to be *strictly convex* if, whenever $x' \succsim_a x$ and $0 < \lambda < 1$, then $\lambda x + (1 - \lambda) x' \succ_a x$. Preferences are said to be *monotone* if:

(i) $x \in X(a)$ and $x' \geq x$ imply $x' \succsim_a x$

(ii) $x \in X(a)$ and $x' \gg x$ imply $x' \succ_a x$.

Preferences are said to be *locally non-satiated* if, given any $x \in X(a)$ and any neighbourhood N of x, there exists x' in N for which $x' \succ_a x$.

3. Limit theorem for replica economies

3.A The m-core

Let $\boldsymbol{E}$ be any economy with a finite set of agents A. A coalition $C \subset A$ *blocks* an allocation $f : A \to X$ in $\boldsymbol{E}$ if there exists $g : C \to X$ such that:

(i) $g(a) \in X(a)$ and $g(a) \succsim_a f(a)$ (all $a \in C$)

(ii) $g(a) \succ_a f(a)$ for some $a \in C$

(iii) $\sum_{a \in C} [g(a) - e(a)] \leq 0$.

An allocation $f : X \to A$ is in the *core* $C(\boldsymbol{E})$ of $\boldsymbol{E}$ if there is no coalition C which blocks f. An allocation $f : A \to X$ is in the *first*

m-*core* $C^1_m(\boldsymbol{E})$ of $\boldsymbol{E}$ if there is no coalition C with $\#C \leq m$ which blocks f. An allocation $f : A \to X$ is m-*feasible* if there is a partition $\boldsymbol{P}$ of A into disjoint sets such that:

(i) $P \in \boldsymbol{P}$ implies $\#P \leq m$

(ii) $\sum_{a \in P} [f(a) - e(a)] \leq 0$ (all $P \in \boldsymbol{P}$).

An allocation $f : A \to X$ is in the m-*core* $C_m(\boldsymbol{E})$ of $\boldsymbol{E}$ if $f \in C^1_m(\boldsymbol{E})$ and also f is m-feasible.

3.B Replica economies

Let $\boldsymbol{E}$ be any economy with a finite set A of N agents. The r^{th} *replica* $\boldsymbol{E}^r$ of $\boldsymbol{E}$ has rN agents in the set A^r who are labelled by the pairs (a, i) with $a \in A$ and $i \in \{1, 2, \ldots, r\}$. For each $a \in A$, all the r agents (a, i) ($i = 1$ to r) share the same preference relation $\succsim_a$, endowment $e(a)$ and feasible set $X(a)$.

3.C An equal treatment property for the m-core

An allocation $f(a, i)$ in the replica economy $\boldsymbol{E}^r$ will be said to have the *equal treatment property*, or to be *symmetric*, if it satisfies $f(a, i) = f(a, j)$ for $i, j = 1$ to r.

Lemma. *Suppose that preferences are strictly convex, and that* $m \geq N$. *Then any allocation* $f(a, i)$ *in the first* m*-core* $C^1_m(\boldsymbol{E}^r)$ *of any replica economy is symmetric and in fact* $C_m(\boldsymbol{E}^r) = C^1_m(\boldsymbol{E}^r)$.

Proof. For each $a \in A$, it loses no generality to assume the agents (a, i) are labelled so that $f(a, i) \succsim_a f(a, 1)$ ($i = 1$ to r). Consider the allocation:

$$g(a) := \sum_{i=1}^{r} f(a, i)/r \qquad (\text{all } a \in A).$$

Then $\sum_{a \in A} g(a) \leq \sum_{a \in A} e(a)$ and, because preferences are strictly convex, $g(a) \succsim_a f(a, 1)$ for all $a \in A$, with indifference only if $f(a, i) = f(a, 1)$ ($i = 2$ to r). Thus the coalition $\{(a, 1) | a \in A\}$ of N agents is a blocking coalition unless, for $i, j = 1$ to r, $f(a, i) = f(a, j)$. Thus symmetry must be satisfied by any allocation f in $C^1_m(\boldsymbol{E}^r)$ unless $m < N$.

By definition, $C_m(\boldsymbol{E}^r) \subset C^1_m(\boldsymbol{E}^r)$. But also, if $m \geq N$ and the allocation $f(a, i) \in C^1_m(\boldsymbol{E}^r)$, we have just seen that f must be symmetric. So f is m-feasible for the partition of the set of Nr agents into the r coalitions $\{(a, i) | a \in A\}$ ($i = 1$ to r) of size N, because $\sum_{a \in A} [f(a, i) - e(a)] \leq 0$. ∎

Of course, this is essentially just an easy adaptation of Debreu and Scarf (1963, Theorem 2).

Write $f : A \to X$ for any symmetric allocation, corresponding to $f(a,i) = f(a)$ for all $(a,i) \in A^r$ $(r = 1,2,\ldots)$.

3.D A property of m-core allocations

Lemma. (Hillas). *Suppose that preferences are strictly convex, and that the integers m, r, r' satisfy $r'N \leq m \leq rN$ (where N denotes $\#A$). Then $C_m(\boldsymbol{E}^r) \subset C(\boldsymbol{E}^{r'})$ (as sets of symmetric allocations).*

Proof. By Lemma 3.C, the cores $C_m(\boldsymbol{E}^r)$, $C(\boldsymbol{E}^{r'})$ both consist of symmetric allocations. Let $f : A \to X$ be a symmetric allocation for which $f \notin C(\boldsymbol{E}^{r'})$. Then there exists a blocking coalition C in the set $A \times \{1,2,\ldots,r'\}$ and an allocation $g : C \to X$ for which:

(i) $g(a,i) \succsim_a f(a)$ (all $(a,i) \in C$)

(ii) $g(a,i) \succ_a f(a)$ (some $(a,i) \in C$)

(iii) $\sum_{(a,i)\in C} [g(a,i) - e(a)] \leq 0$.

Notice that $C \subset A \times \{1,2,\ldots,r\}$ because $r \geq r'$. Therefore C blocks f in $\boldsymbol{E}^r$. Because $\#C \leq r'N$ and $r'N \leq m$, it follows that $\#C \leq m$. So $f \notin C^1_m(\boldsymbol{E}^r) = C_m(\boldsymbol{E}^r)$. ∎

Corollary. If preferences are strictly convex and $m \geq N$, then any allocation in $C_m(\boldsymbol{E}^r)$ is Pareto efficient.[2]

Proof. By the above lemma, $C_m(\boldsymbol{E}^r) \subset C(\boldsymbol{E})$. The corollary follows because all allocations in the core of $\boldsymbol{E}$ are Pareto efficient. ∎

3.E A limit theorem for the m-core

Theorem. *Let $\boldsymbol{E}$ be an economy whose N agents have strictly convex monotone preferences, and suppose that $e(a) \in$ int $X(a)$ (all $a \in A$). Suppose that $m_r \to \infty$ as $r \to \infty$, with $N \leq m_r \leq rN$, $(r = 1,2,\ldots)$. Then $W(\boldsymbol{E}) = \bigcap_{r=1}^{\infty} C_{m_r}(\boldsymbol{E}^r)$ (as sets of symmetric allocations).*

Proof.

(1) By the usual elementary argument, $W(\boldsymbol{E}) \subset C(\boldsymbol{E}^r)$ for all integers r. Evidently $C(\boldsymbol{E}^r) \subset C_m(\boldsymbol{E}^r)$ for all integers m, r with

[2] This corollary was suggested by Malinvaud (1972, p. 174). It led in turn to John Hillas suggesting the above lemma.

$N \leq m \leq Nr$, because any allocation in $C(\boldsymbol{E}^r)$ is symmetric, and so can be brought about with just coalitions of N ($\leq m$) agents—i.e. it is m-feasible—and because it cannot be blocked, let alone m-blocked. So $W(\boldsymbol{E}) \subset \bigcap_{r=1}^{\infty} C_{m_r}(\boldsymbol{E}^r)$.

(2) By Debreu and Scarf (1963, Theorem 3) (see also Debreu 1983) one has:

$$C(\boldsymbol{E}^1) \supset C(\boldsymbol{E}^2) \supset \ldots \supset C(\boldsymbol{E}^r) \supset C(\boldsymbol{E}^{r+1}) \supset \ldots \supset W(\boldsymbol{E})$$

with $W(\boldsymbol{E}) = \bigcap_{r=1}^{\infty} C(\boldsymbol{E}^r)$. It follows that $W(\boldsymbol{E}) = \bigcap_{r=1}^{\infty} C(\boldsymbol{E}^{k(r)})$ provided $k(r) \to \infty$ as $r \to \infty$.

For each r, define $k(r)$ as the largest integer k satisfying $kN \leq m_r$. Then $C_{m_r}(\boldsymbol{E}^r) \subset C(\boldsymbol{E}^{k(r)})$ by Lemma 3.D. So, given that $m_r \to \infty$, which implies that $k(r) \to \infty$ as $r \to \infty$:

$$\bigcap_{r=1}^{\infty} C_{m_r}(\boldsymbol{E}^r) \subset \bigcap_{r=1}^{\infty} C(\boldsymbol{E}^{k(r)}) = W(\boldsymbol{E}). \qquad \blacksquare$$

4. Theorems for a general large economy

4.A In an economy $\boldsymbol{E}$, the consumption sets $X(a)$ ($a \in A$) are said to be *uniformly bounded below* if there exists a number $s \in \mathbb{R}_+$ such that, for all $a \in A$ and all $x \in X(a)$:

$$x - e(a) \geq -s\, 1^\ell$$

where 1^ℓ denotes the vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^\ell$. Write $\Delta := \{p \in \mathbb{R}^\ell | p \geq 0, ||p|| = 1\}$. Also, given the price vector p, the preference relation $\succsim_a$ and the commodity vector $\overline{x} \in \mathbb{R}^\ell$, define:

$$w(p, \succsim_a, \overline{x}) := \inf_x \{px | x \succsim_a \overline{x}\}.$$

Theorem. *(Mas-Colell 1979) Let $\boldsymbol{E}$ be any economy in which agents' consumption sets are uniformly bounded below and their preferences are locally non-satiated. Let $f: A \to X$ be any first m-core allocation in $C_m^1(\boldsymbol{E})$. Then there exists a price vector $p \in \Delta$ such that:*

(i) $(1/\#A) \sum_{a \in A} |p[f(a) - e(a)]| \leq D$

(ii) $(1/\#A) \sum_{a \in A} |w(p, \succsim_a, f(a)) - pe(a)| \leq D$

where $D := 2[(1/m) - (1/\#A)]\ell\, s$. ■

4.B **Theorem.** *(cf. Anderson 1978, Theorem 2) Let $\boldsymbol{E}_n : A_n \to \Theta$ $(n = 1, 2, \ldots)$ be an infinite sequence of economies in which agents'* *consumption sets are uniformly bounded below (as in 4.A) for a sequence of numbers* $s_n \in \mathbb{R}_+$ $(n = 1,2,\ldots)$. *Suppose that* $m_n \leq \#A_n$ *and that* s_n/m_n *converges to zero as* n *tends to infinity. Let* $f_n : A_n \to X$ *be any sequence of first* m_n*-core allocations—i.e.* $f_n \in C^1_{m_n}(E_n)$ $(n = 1,2,\ldots)$. *Then there exists a corresponding sequence of prices* p_n $(n = 1,2,\ldots)$ *in* Δ *such that, as* $n \to \infty$:

(i) $(1/\#A_n)\sum_{a\in A_n} |p_n[f_n(a) - e(a)]| \to 0$

(ii) $(1/\#A_n)\sum_{a\in A_n} |w(p_n, \succ_a, f_n(a)) - p_n e(a)| \to 0$. ■

The theorem says that a sequence of prices p_n can be chosen so that:

(i) the mean absolute deviation from the budget hyperplane $p_n x = p_n e(a)$ shrinks to zero

(ii) the mean absolute deviation from $p_n e(a)$ in the infimum net expenditure required to allow agent a to reach an allocation preferred to $f_n(a)$ shrinks to zero.

Theorem 4.B is an obvious generalization for the m-core of Hildenbrand's lemma (1982, p. 847) which is the basis of all his later analysis. Thus it seems that all his limit theorems can be generalized to the m-core, provided that, in a sequence of economies, both m_n and $\#A_n$ tend to infinity as $n \to \infty$, while s_n/m_n converges to zero.

4.C *If* it were true that $\#A_n \to \infty$, $p_n \to p$ and f_n converges to f in a suitably strong sense, one could conclude from (i) and (ii) that, for almost all agents in a limit economy with $\#A = \infty$, $pf(a) = pe(a) = w(p, \succsim_a, f(a))$ so that one would have a 'compensated Walrasian equilibrium' (*cf.* Kannai 1970, Theorem A). The above 'if', however, is a serious qualification, as can be seen from Hildenbrand (1982) or Mas-Colell (1985, pp. 294–5) for example.

Formally, consider an infinite sequence E_n $(n = 1,2,\ldots)$ of economies with sets of agents $A_n = \{1,2,\ldots,\#A_n\}$ where $\#A_n \to \infty$ as $n \to \infty$. For each such economy E_n, define the *continuum representation of* E_n as the economy E^*_n with the set of agents $A = [0,1)$ and the mapping $E^*_n : A \to \Theta$ from agents to their characteristics given by:

$$E^*_n(a) := E_n(\alpha_n(a)) \quad (\text{all } a \in A)$$

where, for every $a \in [0,1)$, $\alpha_n(a)$ is the unique integer of the set A_n for which

$$\alpha_n(a) - 1 \leq a \cdot \#A_n < \alpha_n(a).$$

Thus $E_n^*(a)$ is a function with (at most) $\#A_n$ steps. Similarly, given any allocation $f_n : A_n \to \mathbb{R}^\ell$ in E_n, define the *continuum representation of* f_n as the allocation $f_n^* : A \to \mathbb{R}^\ell$ in E_n^* with:

$$f_n^*(a) = f_n(\alpha_n(a)) \quad (\text{all } a \in A).$$

The following theorem will be stated for specific economic environments in which each individual agent has a (complete and transitive) preference ordering $\succsim$ which is continuous and also monotone in the sense defined in §2. Following Hildenbrand (1974), the space of such preferences will be given the topology of closed convergence.

Theorem. *Suppose that:*

(1) E_n $(n = 1, 2, \ldots)$ *is an infinite sequence of economies in which agents have preference orderings which are continuous and monotone,*

(2) $\#A_n \to \infty$, *and the continuum representations* E_n^* *converge* a.e. *to the function* $E^* : A \to \Theta$, *where* $A := [0, 1)$,

(3) $f_n : A_n \to \mathbb{R}^\ell$ $(n = 1, 2, \ldots)$ *is an infinite sequence of allocations satisfying* $f_n \in C^1_{m_n}(E_n)$ *whose continuum representations* f_n^* *converge in measure to the function* $f^* : A \to \Theta$,

(4) in each economy E_n *there exists* $s \geq 0$ *such that, for all* $n = 1, 2, \ldots$, *for all* $a \in A_n$ *and all* $x \in X_n(a)$, $x - e_n(a) \geq -s1^\ell$,

(5) $m_n \to \infty$ as $n \to \infty$.

Then there exists a price vector $p^* > 0$ *such that, for almost all* $a \in A$, $p^* f^*(a) = p^* e^*(a)$ *and* $x \succsim_a^* f^*(a)$ *implies* $p^* x \geq p^* e^*(a)$.

The proof of this theorem relies on a crucial lemma. To state it, first define the set $\psi(a) := \{x | x \succsim_a^* f^*(a)\}$ for each $a \in A$, and then $\Psi := \text{co} \int_A \psi d\mu$, where μ is Lebesgue measure on $[0, 1)$ and co denotes the convex hull.

Lemma. $\qquad 0 \notin \text{int}\, \Psi$.

Proof. The result corresponds to the proposition of Kannai (1970, p. 801). The proof here is virtually identical, up to the construction on page 802 of a blocking coalition.[3] Instead of choosing n large enough so that the number of agents $\#A_n$ in the entire economy E_n can accommodate the particular blocking coalition he constructed,

[3] Kannai assumes that $X(a) = \mathbb{R}_+^\ell$ (all $a \in A$), in effect. Assumption 4 of Theorem 4.C substitutes here for Kannai's assumption.

n must be chosen large enough so that m_n exceeds the size of this coalition—as is possible given hypothesis (5) of the Theorem. ■

Proof of Theorem. This is now standard. One can separate the convex set Ψ from the strictly negative orthant, $\mathbb{R}^{\ell}_{--}$ because if the two were to intersect one would have $0 \in \text{int } \Psi$. The separating hyperplane determines prices $p^* \in \Delta$ which serve as required. ■

4.D A difficulty

There is one difficulty, however, which should be noted here. In 3.E, the equal treatment property of the core in replica economies guaranteed that, for $m \geq N$, $W(\boldsymbol{E}) \subset C(\boldsymbol{E}^r) \subset C_m(\boldsymbol{E}^r)$, and in particular that $C_m(\boldsymbol{E})$ is non-empty. Here, however, there is no guarantee that $W(\boldsymbol{E}_n) \subset C_m(\boldsymbol{E}_n)$, because arranging an equilibrium allocation may require coalitions of size greater than m to form. All we know is that $W(\boldsymbol{E}_n) \subset C^1_m(\boldsymbol{E}_n)$.

5. The f^*-core of a continuum economy

5.A Measure-consistent partitions

Let $(S_1, \boldsymbol{S}_1, \sigma_1)$ and $(S_2, \boldsymbol{S}_2, \sigma_2)$ be two measure spaces with $\sigma_1(S_1) = \sigma_2(S_2)$. The measurable mapping $\phi : S_1 \to S_2$ is a *measure-preserving isomorphism* if ϕ^{-1} is a well-defined measurable function, $S_2 = \phi(S_1)$, $S_1 = \phi^{-1}(S_2)$, and if for every $B \in \boldsymbol{S}_1$, $\sigma_1(B) = \sigma_2(\phi(B))$.

Let $C \subset A$ be any coalition. A partition $\boldsymbol{P}$ of C into the family $P_m(\alpha)$ ($m = 1, 2, \ldots$; $0 \leq \alpha \leq \mu_m$) of finite coalitions is said to be *measure-consistent* if there exist:

(i) a partition $\bigcup_{m=1}^{\infty}[\bigcup_{j=1}^{m} C_{mj}]$ of C into a countable collection of measurable subsets with the property that $\mu(C_{mj}) = \mu_m$, independent of j, for $j = 1$ to m; and

(ii) measure-preserving isomorphisms $\phi_{mj} : [0, \mu_m] \to C_{mj}$ ($m = 1, 2 \ldots$; $j = 1$ to m);

such that $P_m(\alpha) = \{\phi_{mj}(\alpha) | j = 1 \text{ to } m\}$ ($m = 1, 2, \ldots ; 0 \leq \alpha \leq \mu_m$).

Here C_{mj} is the set of 'j^{th} members' of coalitions of size m. For any m such that $\mu_m > 0$, there is a measurable continuum $P_m(\alpha)$ ($0 \leq \alpha \leq \mu_m$) of coalitions of size m, such that the measure of the set of j^{th} members is μ_m, independent of j.

The following lemma is used later in the course of proving Theorem 5.E. It follows from Royden (1968, Theorem 9, p. 327); see also Kaneko and Wooders (1986, p. 129).

Lemma. *Suppose that A is a complete separable metric space of agents, and that $\boldsymbol{A}$ is the Borel σ-algebra on A. Let $C \in \boldsymbol{A}$ be any coalition with $\mu(C) = \overline{\mu}$. Then there exist null sets $C^{\circ} \subset C$ and $N \subset [0, \overline{\mu}]$ and a measure-preserving isomorphism $\phi : [0, \overline{\mu}] \setminus N \to C \setminus C^{\circ}$.* ∎

5.B The A-core

Let $\boldsymbol{E} : A \to \Theta$ be a continuum economy, with a non-atomic measure space of agents $(A, \boldsymbol{A}, \mu)$. Let $C \in \boldsymbol{A}$ be a coalition: then C is said to *A-block* an allocation $f : A \to X$ if there exists a measurable function $g : C \to X$ such that:

(i) $g(a) \succ_a f(a)$ a.e. in C

(ii) $\int_C [g(a) - e(a)] d\mu \leq 0$

(iii) $\mu(C) > 0$.

The *A-core* of the economy $\boldsymbol{E}$ (usually called the core, as in Aumann (1964) and many succeeding works) consists of all allocations which are not blocked by any coalition C of the σ-algebra $\boldsymbol{A}$. Write $C_A(\boldsymbol{E})$ for the A-core of the economy $\boldsymbol{E}$.

5.C A Standard elementary result

Lemma. *$W(\boldsymbol{E}) \subset C_A(\boldsymbol{E})$ for every economy $\boldsymbol{E}$.*

Proof. Suppose $f \in W(\boldsymbol{E})$ but C blocks f. Then there are prices p as in §2 and a measurable function $g : C \to X$ as in 5.B for which $g(a) \succ_a f(a)$ a.e. in C. Thus $pg(a) > pe(a)$ a.e. in C, so $p \int_C [g(a) - e(a)] d\mu > 0$. But $p > 0$ and $\int_C [g(a) - e(a)] d\mu \leq 0$, so we have a contradiction. ∎

5.D The f^*-core

A coalition C is said to *f^*-block* an allocation $f : A \to X$ if there exists a measure-consistent partition $\boldsymbol{P}$ of C into finite coalitions and a measurable function $g : C \to X$ such that:

(i) $g(a) \succ_a f(a)$ a.e. in C

(ii*) for all $P \in \boldsymbol{P}$, $\sum_{a \in P} [g(a) - e(a)] \leq 0$

(iii) $\mu(C) > 0$.

Thus, part (ii) of the definition of A-blocking has been strengthened to require exchange among the members of C to take place in a way that makes every finite coalition P of the partition $\boldsymbol{P}$ rely only on its own resources.

An allocation $f : A \to X$ is *f^*-feasible* if there is a measure-consistent partition $\boldsymbol{P}$ of A into finite coalitions such that, for each finite coalition P of $\boldsymbol{P}$:

$$\sum_{a \in P} [f(a) - e(a)] \leq 0 \, .$$

One ought probably to define the f^*-core of an economy $\boldsymbol{E}$ as the set of all f^*-feasible allocations which are not f^*-blocked. But if one does so, the result may well be an empty f^*-core, as shown by Example 2.1 of HKW. Accordingly, I use Kaneko and Wooders (1986, Lemma 3.1) and consider the *closure* (in the topology of convergence in measure) of the set of f^*-feasible allocations. This they show to be equal to the set of all allocations satisfying the usual resource balance constraint of §2:

$$\int_A (f - e) \leq 0 \, .$$

Thus, in the following, one can omit the phrase 'in the closure of the set of f^*-feasible allocations' because this imposes no restriction at all on the allocation f. Accordingly, the *f^*-core* of the economy $\boldsymbol{E}$, denoted by $C_f^*(\boldsymbol{E})$, is defined as the set of those allocations which are not f^*-blocked. Any allocation which is f^*-blocked is clearly A-blocked since (ii*) above implies (ii) of §4.A, so it follows that:

Lemma. *$C_A(\boldsymbol{E}) \subset C_f^*(\boldsymbol{E})$ for every economy $\boldsymbol{E}$.* ■

5.E An equivalence theorem

Theorem. *Suppose that $\boldsymbol{E}$ is a continuum economy, in which preferences are monotone and $X(a)$ is closed and convex, with $e(a) \in$ int $X(a)$ (all $a \in A$). Then:*

$$W(\boldsymbol{E}) = C_A(\boldsymbol{E}) = C_f^*(\boldsymbol{E}) \, .$$

Proof *(cf. Theorem 1 of HKW).*

(1) In view of Lemmas 5.C and 5.D, it is enough to show that $C_f^*(\boldsymbol{E}) \subset W(\boldsymbol{E})$ under the stated hypotheses.

(2) (*cf.* Hildenbrand 1974, Theorem 1, p. 133). For any allocation f, define:

$$\psi(a) := \{x \in \mathbb{R}^\ell \,|\, x + e(a) \succ_a f(a)\} \cup \{0\} \quad (\text{all } a \in A) \, .$$

(3) Suppose that $z \in \int_A \psi$ and $z \ll 0$. Then there exists a measurable subset S of A and a measurable function $t : S \to X$ for which:

(i) $t(a) + e(a) \succ_a f(a)$ a.e. in S

(ii) $\int_S t \ll 0$

(iii) $\mu(S) > 0$.

(4) For $n = 1, 2, \ldots$ define the subset K_n of the interval $[-n, n)$ in $\mathbb{R}$ as the set of rational numbers of the form $k.2^{-n}$ for k an integer satisfying $-n.2^n \leq k < n.2^n$.

(5) Let $1^\ell := (1, 1, \ldots, 1) \in \mathbb{R}^\ell$. Define $t_n(a)$ $(n = 1, 2, \ldots; a \in S)$ as follows:

(i) if $-n1^\ell \leq t(a) \ll n1^\ell$ define $t_n(a)$ so that $2^n t_n(a)$ is the smallest ℓ-vector of integers in Z^ℓ satisfying: $2^n t(a) \ll 2^n t_n(a)$,

(ii) otherwise $t_n(a) := 0$.

(6) Let $\overline{n}(a) := \min\{n \in \mathbb{N} \mid -n1^\ell \leq t(a) \ll n1^\ell\}$. Then $t_n(a)$ $(n \geq \overline{n}(a))$ is a non-increasing sequence of vectors bounded below by $t(a)$, and in fact $t_n(a) \to t(a)$ as $n \to \infty$. Also $\int_S t_n$ converges to $\int_S t \ll 0$ as $n \to \infty$. So there exists a finite n^* such that $\int_S t_{n^*} \ll 0$.

(7) Write $K := K^\ell_{n^*}$, a subset of $\mathbb{R}^\ell$, for the ℓ-fold product of the sets K_{n^*}. For every k in K, define:

$$S(k) := \{a \in S \mid k \leq t(a) \ll k + 2^{-n^*} 1^\ell\}.$$

Then:
$$\int_S t_{n^*} = \sum_{k \in K} (k + 2^{-n^*} 1^\ell)\mu(S(k))$$

by (4) and (5), because $t_{n^*}(a) = 0$ for all $a \in S \setminus [\bigcup_{k \in K} S(k)]$.

(8) Using (6) and (7), for each k in K there exists a subset $S^*(k)$ of $S(k)$ such that $\mu(S^*(k))$ is a rational number and:

$$\sum_{k \in K} (k + 2^{-n^*} 1^\ell)\mu(S^*(k)) \ll 0.$$

(9) By (8), there is a refinement S_j ($j = 1$ to m) of the finite partition $\{S^*(k) \mid k \in K, \mu(S^*(k)) > 0\}$ such that $\mu(S_j) = \overline{\mu} > 0$ ($j = 1$ to m).

(10) Because the m sets S_j ($j = 1$ to m) have equal positive measure, we can discard null subsets S_j^0 of each, if necessary, and

also subsets N_j which are null in $[0,\bar{\mu}]$, after which there will exist measure-preserving isomorphisms $\phi_j : [0,\bar{\mu}] \to S_j$ ($j = 1$ to m), by Lemma 5.A.

(11) By (5), (7), (8) and (9), for $j = 1$ to m there is a constant $\bar{t}_j$, which is equal to $k + 2^{-n^*} 1^\ell$ for the unique $k \in K$ with the property that $S_j \subset S^*(k)$, such that $t_{n^*}(a) = \bar{t}_j$ for all $a \in S_j$.

(12) Define $C := \bigcup_{j=1}^m S_j$. Then, by (8), (9) and (11):

$$\int_C t_{n^*} = \sum_{k \in K} (k + 2^{-n^*} 1^\ell)\mu(S^*(k)) = \sum_{j=1}^m \bar{t}_j \bar{\mu} \ll 0$$

so that $\sum_{j=1}^m \bar{t}_j \ll 0$.

(13) By (3), (5) and (11), for all $a \in S_j$ one has:

$$\bar{t}_j = t_{n^*}(a) \gg t(a) \quad \text{and} \quad t(a) + e(a) \succ_a f(a).$$

So $t_{n^*}(a) + e(a) \succ_a f(a)$ for all $a \in C$, because preferences are monotone.

(14) By (10), (11), (12) and (13), the coalition C f^*-blocks the allocation f, with $g(a) := t_{n^*}(a)$ (all $a \in C$) and with the measure-consistent partition $P := \{P(\alpha) | 0 \leq \alpha \leq \bar{\mu}\}$ where $P(\alpha) := \{\phi_j(\alpha) | j = 1 \text{ to } m\}$ for all $\alpha \in [0,\bar{\mu}]$. So, if there exists $z \in \int_A \psi$ with $z \ll 0$, then $f \notin C_f^*(E)$.

(15) Therefore, if $f \in C_f^*(E)$ the sets $\int_A \psi$ and $\{z | z \ll 0\}$ are disjoint. They are non-empty and convex (using Lyapunov's theorem to establish convexity of $\int_A \psi$, because $(A, \mathcal{A}, \mu)$ is a non-atomic measure space). Thus, there exists a price vector $p \neq 0$ such that:

(i) $pz \geq 0$ for all $z \in \int_A \psi$

(ii) $pz \leq 0$ for all $z \ll 0$.

It follows in particular that $p > 0$.

(16) The rest of the proof follows that of Hildenbrand (1974, Theorem 1, p. 133), for example. ∎

6. The f-core of a continuum economy

6.A The f-core

A *finite* coalition F, a subset of A, is said to *f-block* an allocation $f : A \to X$ if there exist consumption vectors $g(a)$ ($a \in F$) such that:

(i) $g(a) \succ_a f(a)$ (all $a \in F$)

(ii) $\sum_{a \in F} [g(a) - e(a)] \leq 0$.

As usual in defining the core of a continuum economy, f-blocking should be irrelevant if it ceases to be possible after excluding a null set of agents. So an allocation which is f-blocked by some finite coalition is *ineffectively f-blocked* if there is a null subset A^o of A such that every f-blocking coalition intersects A^o. On the other hand, an allocation is *effectively f-blocked* if, for every null subset A^o of A, there exists at least one finite coalition $F \subset A \setminus A^o$ which f-blocks that allocation. Again, as in 5.D, one ought to consider f-feasible allocations, whose definition coincides with that of f^*-feasible allocations, being those allocations can be realized in a measure-consistent partition of finite coalitions. The same difficulty arises, of course, with the same resolution—one considers the closure of the set of f-feasible allocations. Accordingly, the following definition is adopted. The f-core of the economy $\boldsymbol{E}$, denoted by $C_f(\boldsymbol{E})$, consists of those allocations which are not effectively f-blocked.

Evidently, any allocation which is f^*-blocked is effectively f-blocked, so that:

Lemma. *$C_f(\boldsymbol{E}) \subset C_f^*(\boldsymbol{E})$ for every economy $\boldsymbol{E}$.* ■

6.B New equivalence theorem

Theorem. *Suppose that $\boldsymbol{E}$ is a continuum economy, in which preferences are monotone and $X(a)$ is closed and convex with $e(a) \in \text{int } X(a)$ (all $a \in A$). Then:*

$$C_f(\boldsymbol{E}) = C_A(\boldsymbol{E}) = C_f^*(\boldsymbol{E}) = W(\boldsymbol{E}).$$

Proof. By 6.A and 5.E, one has $C_f(\boldsymbol{E}) \subset C_f^*(\boldsymbol{E}) = C_A(\boldsymbol{E}) = W(\boldsymbol{E})$. Here it will be shown that $W(\boldsymbol{E}) \subset C_f(\boldsymbol{E})$.

Suppose that $f \in W(\boldsymbol{E})$, with $p > 0$ as the Walrasian equilibrium price vector. Define the set of agents in Walrasian equilibrium as

$$A^W := \{a \in A \mid f(a) \in X(a),\ pf(a) \leq pe(a),$$
$$\text{and } x \succ_a f(a) \text{ implies } px > pe(a)\}.$$

Let A^o be the complement of A^W. Then $\mu(A^o) = 0$. Also, because preferences are locally non-satiated, a standard elementary argument shows that $pf(a) = pe(a)$ for all $a \in A^W$.

Let F be any finite coalition which f-blocks the Walrasian allocation f with consumption vectors $g(a)$ $(a \in F)$ as in 6.A. Then, for all $a \in F \cap A^{W}$, $pg(a) > pf(a) = pe(a)$. But $\sum_{a \in F} [g(a) - e(a)] \leq 0$ and $p > 0$, so that:

$$\sum_{a \in F} pg(a) \leq \sum_{a \in F} pe(a).$$

This would give a contradiction if $F \subset A^{W}$, so it must be true that F intersects the null set A^{o} whenever F is a coalition which f-blocks the Walrasian allocation.

This proves that $W(\boldsymbol{E}) \subset C_f(\boldsymbol{E})$. ∎

Of course, $W(\boldsymbol{E}) \subset C_f(\boldsymbol{E})$ is actually true even if preferences are just locally non-satiated. The other assumptions of the Theorem are not required for this inclusion.

6.C Widespread externalities

Let F denote the set of allocations $f : A \to X$. In HKW, an *economy with widespread externalities* is defined as one in which:

(i) each agent's consumption set $X(a, f)$ depends in general upon the allocation f in F, with $X(a, f) = X(a, g)$ whenever $f = g$ a.e. in A,

(ii) each agent's preference relation $\succsim_a$ is defined upon the set of pairs (x, f) in $X \times F$ satisfying $x \in X(a, f)$, with (x, f) indifferent to (x, g) whenever $f = g$ a.e. in A.

Thus each individual can be affected by 'widespread' changes in the whole allocation, but must be unaffected by changes to the allocation to other agents in a set of measure zero. Also, each individual or finite coalition can ignore the effect of its own actions on the externalities.

In such an economy, with external diseconomies the A-core can easily be empty. Or, if there are external economies, it consists only of Pareto efficient allocations, naturally. The f-core, however, as defined below, is non-empty in a wide class of models in which what concerns each agent is a finite collection of integrals derived from f, e.g. $\int_A f d\alpha_i$ $(i = 1 \text{ to } n)$, for α_i an absolutely continuous measure on A. A special case is when, for all measurable B and for a finite collection of sets A_i $(i = 1 \text{ to } n)$, $\alpha_i(B) = \mu(A_i \cap B)$. Nor, in general, are allocations in the f-core Pareto efficient. Such inefficiency is right for a concept of perfect competition in the presence of externalities.

In this economy with widespread externalities, a finite coalition $C \in A$ is said to *f-improve* the allocation f if there is an allocation $g : A \to X$ such that:

(i) $g(a) = f(a)$ a.e. in $A \setminus C$

(ii) $(g(a), f) \succ_a (f(a), f)$ (all $a \in C$)

(iii) $\sum_{a \in C}[g(a) - f(a)] \leq 0$.

Notice that this is *not* the usual definition of blocking, in which the complementary coalition $A \setminus C$ attempts to punish the members of C as they seek to block f. Rather, it is closer to the kind of conjecture underlying strong Nash equilibria, in which $A \setminus C$ is assumed to do nothing in response to C's attempts to improve their allocation, as in (i) above. On the other hand, (iii) implies that, as with exchange economies, the coalition C relies only on its own resources; it is effectively presumed that other agents refuse to trade with the members of C.

Having defined f-improvements, an allocation is said to be in the *f-core* $C_f(\boldsymbol{E})$ of the economy $\boldsymbol{E}$ if no coalition can f-improve it. Then letting $W(\boldsymbol{E})$ denote the set of 'Nash–Walrasian equilibria,' in which each individual agent takes widespread externalities as fixed in recognition of the fact that he has no power to affect them, one has equivalence results for the f-core as in 6.A—namely $C_f(\boldsymbol{E}) = W(\boldsymbol{E})$. There is no such equivalence for the f^*-core or the A-core, of course.

Kaneko and Wooders (1989) present a limit theorem like that in §4 for the f-core of an economy with widespread externalities.

6.D Multilateral incentive compatibility

Hammond (1979, 1983) defines a *symmetric allocation* as a mapping $f : \Theta \to X$ from the space of characteristics to the commodity space which satisfies $f(\theta) \in X(\theta)$ for all $\theta \in \Theta$, where $X(\theta)$ denotes the feasible set of a θ-agent. When there is a variable continuum economy $\boldsymbol{E}(\nu)$ with agents in $(A, \boldsymbol{A}, \mu)$, a *symmetric allocation mechanism* $f(\nu, \theta)$ is a mapping $f : M(\Theta) \times \Theta \to X$ defined on pairs consisting of (frequency) distributions ν of characteristics and of characteristics.

Such a mechanism is (straightforwardly) *individually incentive compatible* in the economy $\boldsymbol{E}(\nu)$ if there is *no* pair θ, $\eta \in \Theta$ for which $f(\nu, \eta) P(\theta) f(\nu, \theta)$, where now $P(\theta)$ denotes the strict preference relation of a θ-agent. This restricts coalitions to single agents, in effect. Gale (1980, 1982) considered finite coalitions in sequence economies and, presuming Pareto efficiency, was able to prove that multilateral incentive compatibility ('strict Nash equilibrium') required a Walrasian allocation mechanism in many sequence economy environments. Hammond (1983) proves more general results for static economies without assuming Pareto efficiency. Both of us gave finite coalitions the power to exchange goods, as in the f-core, as well as to misrepresent their true characteristics.

When all goods are exchangeable, the symmetric allocation $f(\theta)$ is said to be *multilaterally incentive compatible* if there is no continuum economy $\boldsymbol{E}$ with a non-atomic measure space $(A, \boldsymbol{A}, \mu)$ of agents, no finite coalition $C \subset A$ with true characteristics θ_a $(a \in C)$, and no combination of characteristics η_a and net trades t_a such that:

(i) $f(\eta_a) + t_a P(\theta_a) f(\theta_a)$ (all $a \in C$)

(ii) $\sum_{a \in C} t_a \leq 0$.

When agents cannot misrepresent their true characteristics, this reduces to f-blocking. So it is not too surprising that multilateral incentive compatibility requires allocations to be Walrasian in the special case when all goods are exchangeable. When C consists of a single individual, the above definition reduces to individual incentive compatibility.

Notice that even *individuals* have power in continuum economies to upset incentive incompatible mechanisms. If one agent can gain by claiming a false characteristic, it is almost certain that a continuum of agents with nearby characteristics can make similar gains.

7. Concluding remarks

This paper has shown that coalitions whose relative size shrinks to zero as the economy grows may still have the power to make the core collapse to the set of Walrasian allocations in the limit, and that even in continuum economies there are consistent and useful ways of modelling the power that finite coalitions have (even if it takes a large number of small coalitions acting independently to have a noticeable effect).

While much interesting work has dealt with largish coalitions in large economies, we claim that our approach is both logically consistent and able to produce new results. Indeed, there seems reason to believe that competitive firms, stock markets, etc. may be much more amenable to analysis based upon small coalitions in large economies.[4]

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[4] It may also turn out that the f-core of an infinite horizon overlapping generation economy, with a continuum of agents in each generation, as in Samuelson (1958), is equivalent to the set of Walrasian allocations, even though these are Pareto inefficient and so not in the A-core. See Chae (1987) for results which are closely related. I am grateful to Karl Shell for this reference.

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